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"Using Matlab to Find the Total Mean Curvature"

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Abstract:

This study aims to calculation the total mean curvature using Matlab .We followed the applied mathematical method using Matlab and we found the following some results: To find the total mean curvature , the First and Second Fundamental Forms of Surfaces must be calculated .Finding the total mean curvature by Matlab is more accurate and fast and can be plotted .

Key Wards: Calculation , Total Mean Curvature , Matlab

1. Introduction:

In daily life, we see many surfaces such as balloons, tubes, tea cups and thin sheets such as soapbubble, which represent physical models. To study these surfaces, we need coordinates for the work of the necessary calculations. These surfaces exist in the triple space, but we cannot think of them as three dimensional. For example, it we cut a cylinder longitudinal section, it can be individual or spread to become a flat ona desktop. This shows that these surfaces are two- dimensional inheritance and this should be described by the coordinates. This gives us the first impression of how the geometric description of the surface .The regular surface can be obtained by distorting the pieces of flat paper and arranging them in such away that the resulting shape is free of sharp dots or pointed letters or intersections (the surface cutsitself) and thus can be talked about the tangent level at the shape points .The surface is said to be in the triple vacuum R^3 is subgroup from R^3 (Le, a special pool of points) of course, not all particle groups are surfaces and certainly mean smooth and two – dimensional surfaces.

Matlab is a software package for computation in engineering, science, and applied mathematics. It offers a powerful programming language excellent graphics and a wide range of expert knowledge. Matlab is published by a trademark of the math works.The focus in Matlab is on computation, not mathematics : Symbolic expressions and manipulations are not possible (except through the optional symbolic tool box, a clever interface to Maple). All results are not only numerical but inexact . The limitation to numerical computation can be seen as a drawback, but it is a source of strength too: Matlab is much preferred to Maple, Mathematic, and the like when it comes to numeric .

On the other hand compared to other numerically oriented languages like C^{++} and Fortran, Matlab is much easier to use and comes with a huge standard library. The unfavorable comparison here is a gap in execution speed. This gap is not always as dramatic as popular lore has it, and it can often be narrowed or closed with good Matlab programming. Moreover, one can link other codes into Matlab, or vice versa, and Matlab. (Brian D.Hahn,)

2. Surfaces in R^3:

Definition (2.1): A C^∞ Coordinate chart is a C^∞ map x from an open subset of R^2 into R^3.

$$x : U \in R^2 \longrightarrow R^3$$

$$(u,v) \xrightarrow{x} \left(x(u,v), y(u,v), z(u,v)\right) \quad (2.1)$$

We always assume that the Jacobian of the map has maximal rank. In local coordinates, a coordinate chart is represented by three equations in two variables:

$$x^i = f^i(u^\alpha), \text{ Where } i = 1,2,3 \ \alpha = 1,2 \quad (2.2)$$

The assumption that the Jacobian J= $(\partial x^i/\partial u^\alpha)$ be of maximal rank allows one to invoke the Implicit function Theorem. Thus, in principle, one can locally solve for one of the coordinates, say x^3, in terms of the other two, like so:

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$$x^3 = f\left(x^1, x^2\right) \quad (2.3)$$

The locus of points in R^3 satisfying the equations $x^i = f^i\left(u^\alpha\right)$ can also be locally represented by an expression of the form:

$$F\left(x^1, x^2, x^3\right) = o \quad (2.4)$$ (**Gabriel lugo , 2006)**

Definition (2.2): Let $x\left(u^1, u^2\right): u \longrightarrow R^3$ and $y\left(v^1, v^2\right): v \longrightarrow R^3$ be two coordinate charts with anon – empty intersection $X(u) \cap Y(v) \neq \emptyset$. The two charts are said to be C^∞ equivalent if the map $\varnothing = y^{-1}x$ and its inverse $\varnothing^{-1}$ are infinitely differentiable the definition simply states that two equivalent charts $x(u^\alpha)$ and $y\left(v^\beta\right)$ represent different reparametrizations for the same set of points in R^3 **(Alan wenstin)**

Definition (2.3): A differentiability smooth surface in R^3 is a set of points M in R^3 satisfying the following properties:

- If $p \in M$ then p belongs to some C^∞ chart.

- If $p \in M$ belongs to two different charts x and y, then the two charts are C^∞ equivalent. Intuitively, we may think of a surface as consisting locally of number of patches "sewn" to each other so as to form a quilt from a global perspective. The first condition in the definition states that each local patch looks like apiece of R^2, whereas the second differentiability condition indicates that the patches are joined together smoothly. Another way to state this idea is to say that a surface a space that is locally Euclidean and it has a differentiable structure so that the notion of differentiation makes sense .If the Euclidean space is of dimension n, then the surface is called ann-dimensional manifold.(**JofferyM.lee– 2000**)

Examples (2.4): Consider the local coordinate chart:

$$x(u,v) = (\sin u \cos v,\ \sin u \sin v,\ \cos v)$$

The vector equation is equivalent to three scalar functions in two variables:

$$x = \sin u \cos v,\ y = \sin u \sin v,\ z = \cos u \quad (2.5)$$

Clearly, the surface represented by this chart is part of the sphere $x^2 + y^2 + z^2 = 1$. The chart cannot possibly represent the whole sphere because although a sphere is locally Euclidean,(the earth is locally flat) there is certainly a topological difference between a sphere and a plane. Indeed, if one analyzes the coordinate chart carefully, one will note that at the North pole $(u = 0,\ z = 1)$ the coordinates become singular. This happens because $u = 0$ implies that $x = y = 0$ regardless of the value of v, so that the North pole has an infinite number of labels. The fact that it is required to have two parameters to describe a patch on a surface in R^3 is a manifestation of the 2-dimensional nature of the surfaces. If one holds one of the parameters constant while varying the other, then the resulting 1- parameter equations describe a curve on the surface.Thus, for example, letting $u =$constant in equation (2.5),we get the equation of a meridian great circle. **(Carl Johan lejdfors ،2003) .**

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Notation (2.5): Given a parameterization of a surface in local chart $x(u, \mathrm{v}) = x(u^1, u^2) = x(u^\alpha)$, we denote the partial derivatives by any of the following notations :

$$x_u = x_1 = \frac{\partial x}{\partial u} \quad x_{uu} = x_{11} = \frac{\partial^2 x}{\partial u^2}$$

$$x_v = x_2 = \frac{\partial x}{\partial v} \quad x_{vv} = x_{22} = \frac{\partial^2 x}{\partial v^2}$$

$$x_\alpha = \frac{\partial x}{\partial u^\alpha} \quad x_{\alpha\beta} = \frac{\partial^2 x}{\partial u^\alpha \partial v^\beta}$$ (Dietmar A. Salamon , 2013)

3. The First Fundamental form:

Let $x^i(u^\alpha)$ be a local parameterization of a surface. Then, the Euclidean inner product in R^3 induces an inner product in the space of tangent vectors at each point in the surface. This metric on the surface is obtained as follows:

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha$$

$$dS^2 = \partial_{ij} dx^i dx^j$$

$$= \partial_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} du^\alpha du^\beta$$

Thus, $dS^2 = g_{\alpha\beta}\, du^\alpha du^\beta$ (3.1)

Where $g_{\alpha\beta} \partial_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta}$ (3.2)

We conclude that the surface, by virtue of being embedded in R^3, inherits a natural metric (3.1). Which we call the induced metric. A pair $\{M, g\}$, where M is a manifold and $g = g\, du^\alpha \otimes du^\beta$ is a metric is called a Riemannian manifold if considered as an entity in it self, and Riemannian submanifold of R^n if viewed as an object embedded in Euclidean space. An equivalent version of the metric (3.1) can be obtained by using a more traditional calculus notation:

$$dx = x_u du + x_v dv$$

$$ds^2 = dx.dx$$

$$= \left(x_u du + x_v dv\right).\left(x_u du + x_v dv\right)$$

$$= \left(x_u . x_u\right) du^2 + 2\left(x_u . x_v\right) du dv + \left(x_v . x_v\right) dv^2$$

We can rewrite the last result as

$$ds^2 = E\ du^2 + 2F\ du\ dv + G\ dv^2 \quad (3.3)$$

Where : $E = g_{11} = x_u . x_u$, $F = g_{12} = x_u . x_v$, $= g_{21} = x_v . x_u$, $G = g_{22} = x_v . x_v$

That is : $g_{\alpha\beta} = x_\alpha . x_\beta = < x_\alpha , x_\beta >$ **(Theodore Shifrin , 2016)**

Proposition (3.1) :

The parametric lines are orthogonal if $F = 0$. (Alan wenstin, Joseph Oesterle)

4.The Second Fundamental Form:

Let $x = x\ (u^\infty)$ be a coordinate patch on a surface M . Since x_u and x_v are tangential to the surface, we can construct a unit normal n to the surface by taking

$$n = \frac{x_u \times x_v}{\left\| x_u \times x_v \right\|} \quad (4.1)$$

Now, consider a curve on the surface given by $u^\alpha = u^\alpha\left(s\right)$. Without loss of generality, we assume that the curve is parameterized by arc length s so that the curve has unit speed. Using the chain rule, we see that the unit tangent vector T to the curve is given by

$$T = \frac{dx}{ds} = \frac{dx}{du^\alpha} . \frac{du^\alpha}{ds} = x_\alpha \frac{du^\alpha}{ds} \quad (4.2)$$

Since the curve lives on the surface and the vector T is tangent to the curve, it is clear that T is also tangent to the surface. On the other hand, the vector $T' = dT / ds$ doesn't in general have this property, so we decompose T' in to its normal and tangential components

$$T' = K_n + K_g \quad (4.3)$$

Where $K_n = \| K_n \| = \langle T', n \rangle$.

The scalar quantity k_n is called the normal curvature of the curve and k_g is called the geodesic curvature vector. The normal curvature measures the curvature of $x\left(u^\alpha\left(s\right)\right)$ resulting from the constraint of the curve to lie on a surface. The

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geodesic curvature vector, measures the "Side ward" component of the curvature in the tangent plane to the surface. Thus, if one draws a straight line on a flat piece of paper and then smoothly bends the paper into a surface, then the straight line would now a acquire some curvature. Since the line was originally straight, there is no sideward component of curvature so $k_g = 0$ in this case. This means that the entire contribution to the curvature comes from the normal component, reflecting the fact that the only reason there is curvature here is due to the bend in the surface itself.

Similarly, if one specifies a point $p \in M$ and a direction vector $X_p \in T_pM$, one can geometrically envision the normal curvature considering the equivalence class of all unit speed curves in M that contain the point p and whose tangent vectors line up with the direction of X .Of course, there are infinitely many such curves, but at an infinitesimal level, all these curves can be obtained by intersecting the surface with a "Vertical" plane containing the vector X and the normal to M . All curves in this equivalence class have the same normal curvature and their geodesic curvatures vanish. In this sense, the normal curvature is more of a property pertaining to a direction on the surface at a point, whereas the geodesic curvature really depends on the curve itself. It might be impossible for a hiker walking on the undulating hillsof the Ozarks to find a straight line trail, since the rolling hills of the terrain extend in all directions. It might be possible. However, for the hiker to walk on a path with Zero geodesic curvature as a long the same compass direction is maintained. To find an explicit formula for the normal curvature we first differentiate equation (4.2)

$$T' = \frac{dt}{ds}$$

$$= \frac{d}{ds}\left(x_\alpha \frac{du^\alpha}{ds} \right)$$

$$= \frac{d}{ds}(x_\alpha)\frac{du^\alpha}{ds} + x_\alpha \frac{d^2u^\alpha}{ds^2}$$

$$= \left(\frac{dx_\alpha}{du^\beta} . \frac{du^\beta}{ds} \right)\frac{du^\alpha}{ds} + x_\alpha \frac{d^2u^\alpha}{ds^2}$$

$$= x_{\alpha\beta} \frac{du^\alpha}{ds} . \frac{du^\beta}{ds} + x_\alpha \frac{d^2u^\alpha}{ds^2}$$

Taking the inner product of the last equation with the normal and noticing that $\left\langle x_\mu ,\ n \right\rangle = 0$, We get

$$k_n = \langle T^{\cdot}, n \rangle = \langle x_{\alpha\beta}, n \rangle \frac{du^\alpha}{ds}\frac{du^\beta}{ds}$$

$$= \frac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta} \quad (4.4)$$

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Where:

$b_{\alpha\beta} = \langle x_{\alpha\beta}, n \rangle$ (4.5) (Gabriel lugo ،2006.)

Example(4.1):

Find $\sqrt{\Pi}$ with the command sqrt(pi). The answer should be 1.7725. Note that Matlab knows the value of pi, because it is one of Matlab's many built-in functions. **(Rao.v.Dukkipati ،2004)**

(4.2) Curvature:

An important fact about covariant derivatives is that they don't need to commute. If $\sigma : M \longrightarrow E$ is a section and $X \in x(m)$ then ∇x^{σ} is a section also and so we may take it's covariant derivative $\nabla_y \nabla_{x^\sigma} \neq \nabla_x \nabla_{y^\sigma}$ and this fact has an underlying geometric interpretation which we will explore later.

A measure of this lack of commutatively is the curvature operator which is defined for a pair $x, y \in x(m)$ to be the map $F(x,y) : \Gamma(E) \longrightarrow \Gamma(E)$ defined by

$$F(x,y)\sigma = \nabla_x \nabla_{y^\sigma} - \nabla_y \nabla_{x^\sigma} - \nabla_{[x,y]^\sigma}$$

Or $\left[\nabla_x, \nabla_y\right]\sigma - \nabla_{[x,y]^\sigma}$ (4.5). **(Barret O'Neill ،2006)**

Definition (4.2.1): The lie bracket $[x, y]$ of two vector fields x and y an a surface M is defined as the commentator, $[x, y] = xy - yx$ (4.6)

Meaning that if f is a function on M, then

$[x,y](f) = x(y(f)) - y(x(f))$ (4.7) **(Carl Johan lejdfors ،2003)**

Proposition (4.2.2):

The lie bracket of two vectors $x, y \in T(M)$ is another vector in $T(m)$.

Proof: If suffices to prove that the bracket is a linear derivation on the space of C^{∞} functions consider vectors $x, y \in T(m)$ and smooth function f, g in M, then,

$$[x,y](f+g) = x(y(f+g)) - y(x(f+g))$$

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$$=x\left(y(f)\right)+y(g)-y\left(x(f)\right)+x\left((g)\right)$$
$$=x(y(f))-y(x(f))+x(y(g))-y(x(g))$$
$$=[x,y](f)+[x,y](g)$$

and

$$[x,y](fg)=x\left(y(fg)\right)-y\left(x(fg)\right)$$
$$=x\left[f\,y(g)+g\,y(f)\right]-y\left[f\,x(g)+gx(f)\right]$$
$$=x(f)y(g)+f\,x\left(y(g)\right)+x(g)y(f)+g\,x\left(y(f)\right)-y(f)x(g)$$
$$-f(y(x(g))-y(g)\,x(f)-gy(x(f))$$
$$=f\left[x\left(y(g)\right)-y\left(x(g)\right)\right]+g\left[x\left(y(f)\right)-y\left(x(f)\right)\right]$$
$$=f[x,y](g)+g[x,y](f).$$ (Taha Sochi, 2016)

(4.3) Mean curvature:

Definition (4.3.1): Let K_1 and K_2 be the principal curvatures of a surface path $\sigma(u,v)$. The mean curvature of σ is

$$H=\frac{1}{2}(k_1+k_2)$$

To compute k and H, we use the first and second fundamental forms of the surface:

$$Ed^2u+2F\,du\,dv+G\,d^2v \text{ and } Ld^2u+2m\,du\,dv+N\,d^2v$$

Again, we adopt the matrix notation:

$$F_1=\begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{ and } F_2=\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

By definition, the principal curvatures are the eigenvalues of $F_1^{-1}F_2$. Hence the determinant of this matrix is the product $k_1\,k_2, i.e$, the Gaussian curvature k. So

$$k = \det\left(F_1^{-1}F_2\right)$$

$$= \det\left(F_1^{-1}\right)\det\left(F_2\right)$$

$$= \frac{LN - M^2}{EG - F^2} \quad (4.8)$$

The trace of the matrix is the sum of its eigenvalues thus, twice the mean curvature H. After some calculation, we obtain :

$$H = \frac{1}{2}\operatorname{trace}\left(F_1^{-1}F_2\right)$$

$$= \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2} \quad (4.9)$$ **(Barret O'Neill ،2006)**

(4.4) Total Curvature:

One of the most important geometric invariants of a surface is gotten by integration of curvature.

Definition (4.4.1): Let k be the Gaussian curvature of a compact surface M oriented by are form dM. Then :

$$\iint_M KdM$$

Is the total Gaussian curvature M.

The same definition applies to any payable region M. To compute total curvature of M we add the total curvatures of each patch like 2-segment X of a paving of M. With the usual notation for the domain R of X,

$$\iint_x KdM = \iint_R x^*\left(k\,d\,M\right) = \iint_R K\left(x\right)x^*\left(dM\right)$$

$$= \int_a^b\int_c^d K\left(x\right)\sqrt{EG - F^2}\,du\,dv$$ (Taha Sochi, 2016)

5. Calculation of the Total Mean Curvature Using Matlab:

x	*1*	*2*	*4*	*8*
y	*0.5*	*1*	*1.5*	*2*

$$k_1 = 2x + 3y$$

$$k_2 = -2x + 5y + 4$$

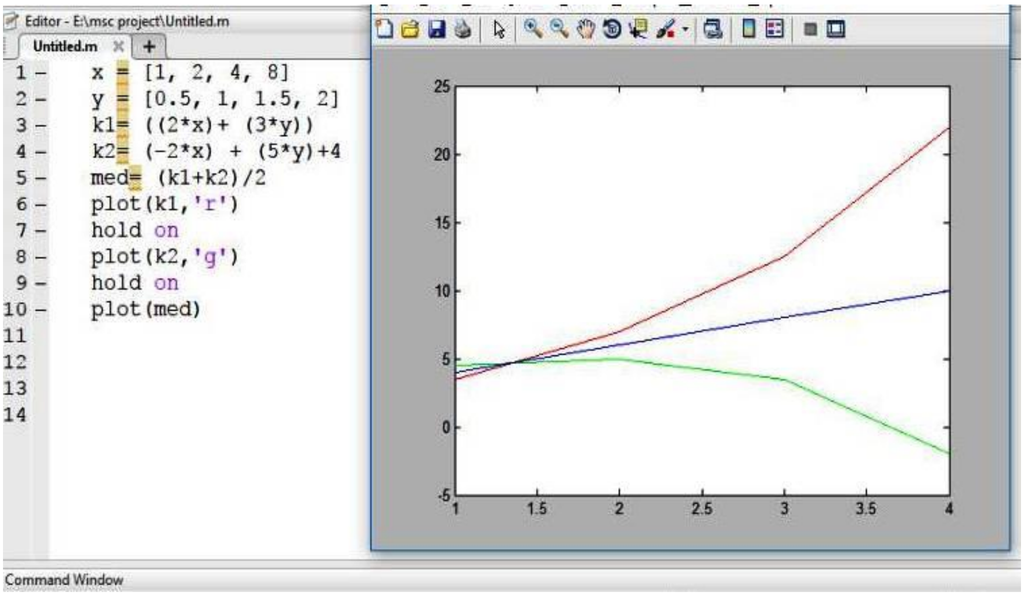

Fig No (5.1):Total Mean Curvature

Fig. No. "1": Calculating the Total Mean Curvature using Matlab

6. Results:

The first fundamental form plays such a crucial role in the theory of surfaces that we found it convenient to introduce a third, more modern version, all curves in equivalence class have the same normal curvature and their geodesic curvatures vanish, the second fundamental form is symmetric, the first fundamental form on a surface measures the square of the distance between two infinitesimally separated points. There is a similar interpretation of the second fundamental form as we show below. The second fundamental form measures the distance from a point on the surface to the tangent plane at a second infinitesimally separated point, to calculate the total mean curvature we used the first and second fundamental forms of the surfaces and we found the possibility of calculating total mean curvature using Matlab

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