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"The Continuous General Solution of Functional equation related to Characteristic of two by two Symmetric Matrices"

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Abstract:

In this paper, we obtained and reached a proof of a continuous general solution $f: R^2 \to R$, to each of the following functional equations:

1. $f(\mathrm{ux} + \mathrm{vy}, \mathrm{uy} + \mathrm{vx}) = f(x,y)f(u,v), x, y, u, v \in R$.
2. $f(\mathrm{ux} - \mathrm{vy}, \mathrm{uy} - \mathrm{vx}) = f(x,y)f(u,v), x, y, u, v \in R$.
3. $f(\mathrm{ux} - \mathrm{vy}, \mathrm{uy} - \mathrm{vx}) = f(x,y)f(u,v) + f(x,y) + f(u,v), x, y, u, v \in R$.
4. $f(\mathrm{ux} + \mathrm{vy}, \mathrm{uy} - \mathrm{vx}) = f(x,y)f(u,v) + f(x,y) + f(u,v), x, y, u, v \in R$.

Which arises from the characterizations of two by two symmetric matrices, the results that we obtained can be applied to deduce other solutions to a number of related functional equations, and the solution can be polynomial function, when arbitrary constant is a natural number.

Keywords: permanent, determinant of matrix, functional equation, general solution, multiplicative function, continuous general solution

1. Introduction:

In mathematics, functional equations are equations in which the unknown (or unknowns) are functions. , and solver's mission is to determine their explicit forms. To solve a functional equation means to find all functions, which satisfy it identically. Functional equations arise in various areas of mathematics, usually when we describe all functions that have a certain properties. One of the famous functional equation that represents our study, known as the power Cauchy functional equation denoted by:

$$f(x,y) = f(x) \cdot f(y). \tag{1}$$

Which plays an important role in geometric objects theory and invariants theory that has been solve and studied in many spaces by many authors. See[1] – [5].

Now let us define a function $f: R^2 \to R$, by $f(x,y) = det\begin{pmatrix} x & y \\ y & x \end{pmatrix}$, For all $x, y \in \mathbb{R}$, now we deal with the famous properties in matrices, states that:

$$det\begin{pmatrix} x & y \\ y & x \end{pmatrix} det\begin{pmatrix} u & v \\ v & u \end{pmatrix} = det\begin{pmatrix} ux+vy & uy+vx \\ uy+vx & ux+vy \end{pmatrix}.$$

Which leads to the following interesting functional equation:

$$f(\mathrm{ux} + \mathrm{vy}, \mathrm{uy} + \mathrm{vx}) = f(x,y)f(u,v) \tag{2}$$

In 2002, Chang and Sahoo [6] have found the general solution of equation(2), which has given by the following function:

$$f(x,y) = M_1(x+y) \cdot M_2(x-y), \tag{3}$$

Where, $M_1, M_2: R \to R$ are multiplicative functions.

In addition, another functional equation arises from the note:

$$det\begin{pmatrix} u & v \\ v & u \end{pmatrix} = det\begin{pmatrix} u & -v \\ -v & u \end{pmatrix}.$$

Thus, we have:

$$det\begin{pmatrix} x & y \\ y & x \end{pmatrix} det\begin{pmatrix} u & -v \\ -v & u \end{pmatrix} = det\begin{pmatrix} ux-vy & uy-vx \\ uy-vx & ux-vy \end{pmatrix}$$

Which leads to the functional equation functional equation:

$$f(ux - vy, uy - vx) = f(x,y)f(u,v) \tag{4}$$

Where its general solution is:

$$f(x,y) = M(x^2 - y^2). \tag{5}$$

Where M: $R \to R$ is a multiplication function. See [7]
Variant of the above equation (4) is the following:

$$f(ux - vy, uy - vx) = f(x,y)f(u,v) + f(x,y) + f(u,v) \tag{6}$$

In 2006, Houston and Sahoo [7] have found the general solution of the equation(6), which had given by the following functions:

$$f(x,y) = M(x^2 - y^2) - 1. \tag{7}$$

Where M: $R \to R$ is a multiplication function.
Similarly, the permanent of a matrix defined by:

$$\text{per}\begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 + y^2$$, matching with the following:

$$per\begin{pmatrix} x & y \\ y & x \end{pmatrix} per\begin{pmatrix} u & v \\ v & u \end{pmatrix} = per\begin{pmatrix} ux + vy & uy - vx \\ uy - vx & ux + vy \end{pmatrix}$$

Leads to an interesting equation as follows:

$$f(ux + vy, uy - vx) = f(x,y) \cdot f(u,v), \tag{8}$$

$x, y, u, v \in R$.which has the following solution:

$$f(x,y) = M(x^2 + y^2), \tag{9}$$

Where M: $R \to R$ is a multiplication function. [7]
A variant of the above equation (8) is the following

$$f\,(ux + vy, uy - vx) = f(x,y)f(u,v) + f(x,y) + f(u,v) \tag{10}$$

Which determined its general solution by: [7]

$$f(x,y) = M(x^2 + y^2) - 1. \tag{11}$$

Where M: $R \to R$ is a multiplication function.
Similarly, we have

$$per\begin{pmatrix} x & y \\ y & x \end{pmatrix} per\begin{pmatrix} v & u \\ u & v \end{pmatrix} = per\begin{pmatrix} \text{vx} + \text{uy} & \text{vy} - \text{ux} \\ \text{vy} - \text{ux} & \text{vx} + \text{uy} \end{pmatrix}$$

If, we define function $f: R^2 \to R$, by $f(x,y) = per\begin{pmatrix} x & y \\ y & x \end{pmatrix}$, then we obtain the following functional equation:

$$f(\text{vx} + \text{uy}, \text{vy} - \text{ux}) = f(x,y) \cdot f(u,v), \quad x, y, u, v \in R\,. \tag{12}$$

In 2017 Laohak osoli, and Suriyacharoen [8] have found the general solution of equation (12) as follows:

$$f(x,y) = M\left(\sqrt{x^2 + y^2}\right)V(\theta_{xy}), \tag{13}$$

Where M: $R \to R$ is also a multiplicative function, V: R→R is a positive-valued exponential function subject to the condition that V (2π) = 1, and θ_{xy} is the angular coordinate of the polar coordinate of (x, y).

In addition, many studies deal with kind of equation from its view of stability, the interesting reader should refer to [9], [10] for an in-depth account of the subject of functional equations see [11] – [18].

Then by the study of these researches, and through investigation in this area, the researcher felt a problem that related to the general solutions of these equations. All this increased the researcher's sense to this study, which prompted him to conduct a research presents new solutions to these previous equations in light of the continuity of functions solution.

2. Problem statement and objective:

General solution of a functional equation related to determinants and permanent of two by two symmetric matrices, have been introduce as a multiplicative function; here we deal with some changes to solutions in the light of continuity. So this paper target:

i. To rework previous general solutions of these equations in the light of continuity behavior of functions.
ii. To investigate an advanced proof of these general solutions in the light of continuity behavior of functions.

Our method solutions of these equations will be simple. These equations have connected with characterizations of the determinant and the permanent of two-by-two symmetric matrices. So our paper classified as descriptive analytical and deductive one, so we use various resources such as journal articles, books, and web sites, so we use this method to answer the following questions:

i. Why the general solution function of functional equation related to characteristic of two by two matrices changed its rule, when it is continuous?
ii. How we can investigate the proof of general solution of functional equation related to the characterizations of two by two symmetric matrices when it is behaving as a continuous function?

Table 1Diagram of sorting solutions

No	General solution	Continuous General solution
(1)	$f(x,y) = M_1(x+y)M_2(x+y).$	$f(x,y) = (x+y)^a(x+y)^b.$
(2)	$(x,y) = M(x^2-y^2).$	$f(x,y) = \|x^2-y^2\|^\alpha.$
(3)	$(x,y) = M(x^2-y^2)-1$	$f(x,y) = \|x^2-y^2\|^\alpha - 1.$
(4)	$f(x,y) = M(x^2+y^2)$	$f(x,y) = (x^2+y^2)^\alpha.$
(5)	$f(x,y) = M(x^2+y^2)-1$	$f(x,y) = (x^2+y^2)^\alpha - 1.$

3. Preliminary results:

In this section, we recall basic facts of functional equation theory, which we need it here in this paper.

Definition 3.1. A function $M : R \to R$ is said to be a multiplicative function if and only if it satisfies $M(xy) = M(x)M(y)$ for all $x, y \in R$. An identically constant multiplicative function M is either $M = 0$ or $M = 1$, which is an arithmetic function [2].

Theorem 3.2. Let $D \subseteq R$ be an interval such that if $x, y \in D$, then $xy \in D$. The general solution $f: D^2 \to R$ of the functional equation

$$f(x_1y_2, x_2y_1) = f(x_1, y_1)f(y_2, x_1) \qquad (14)$$

Holding for all $x_1, y_1, x_2, y_2 \in D$ be given by

$$f(x,y) = M_1(x) \cdot M_2(y), \qquad (15)$$

Where $M_1, M_2 : D \to R$ are multiplicative functions.

Proof. It is easy to check that the solution enumerated in (15) satisfies functional equation(14). Next, we will show that (2.10) is the only solution of (14). Suppose f is identically a constant, say $f \equiv c$. Then from (14) we have $c^2 - c = 0$ for all $x, y \in D$. Hence $c = 1$ or $c = 0$. Thus the only identically constant solutions of (14) are $f(x,y) = 1$ and $f(x,y) = 0$ for all $x, y \in D$, which are solutions included in (15).

From now on, we assume that f is not identically constant. Let $a \in D$ be a fixed element and $f : D^2 \to R$ be such that it satisfies (14) with$f(a,a) \neq 0$. Then

$$f(x,y) = f(x,y)f(a,a)f(a,a)f(a,a)^{-2}$$

$$= f(xaa, aay)f(a,a)^{-2}$$

$$= f((xa)a, (ya)a)f(a,a)^{-2}$$

$$= f(xa,a)f(a,ya)f(a,a)^{-2}$$

$$= f(xa,a)f(a,a)^{-1} f(a,ya)f(a,a)^{-1}$$

$$= M_1(x)M_2(y)$$

Where,

$$M_1(x) := f(xa,a)f(a,a)^{-1}.$$

And,

$$M_2(x) := f(a,ya)f(a,a)^{-1}.$$

Now we show that M_1 and M_2 are multiplicative functions in D. Consider $M_1(xy) := f(xya,a)f(a,a)^{-1}$

$$= f(xya,a)f(a,a)f(a,a)^{-2}$$

$$= f(xyaa, aa)f(a,a)^{-2}$$

$$= f(xa,a)f(ya,a)f(a,a)^{-2}$$

$$= f(xa,a)f(a,a)^{-1} f(ya,a)f(a,a)^{-1}$$

$$= M_1(x)M_1(y).$$

Hence M_1 is multiplicative. Similarly, one can show that

$$M_2(xy) = M_2(x)M_2(y).$$

Hence M_2 is multiplicative. Thus

$$f(x,y) = M_1(x)M_2(y).$$

Where M_1 and M_2 are multiplicative functions.

Theorem 3.3. let $\mathfrak{D} = \mathcal{R} - \{0\}$, if a function $f: \mathfrak{D} \to \mathcal{R}$, satisfies the equation (1),then there exist an additive function $g: \mathcal{R} \to \mathcal{R}$, such that $f(x) = g(log|x|)$.

Theorem 3.4. let f, be a function $f: \mathcal{R} \to \mathcal{R}$, satisfies the equation (1),then there exist an additive function $g: \mathcal{R} \to \mathcal{R}$, such that f has one of the following forms:

$$f = 0, \quad (16)$$

$$f = 1, \quad (17)$$

$$f(x) = \begin{cases} exp(g(log|x|)), x \neq 0 \\ 0, \quad\quad x = 0 \end{cases}, \quad (18)$$

$$f(x) = \begin{cases} exp(g(log|x|)), & x > 0 \\ 0, & x = 0, \\ -exp(g(log|x|)), & x < 0 \end{cases} \quad (19)$$

Theorem 3.5. let $f: \mathcal{R} \to \mathcal{R}$ be a continuous function satisfied the following Cauchy's equation:

$$f(x) + f(y) = f(x + y) \quad (20)$$

For all $x, y \in \mathcal{R}$, then there exists a real number c such that $f(x) = cx$, for all $x \in \mathcal{R}$. For the proof see[18] .

Theorem 3.6. A function$f: \mathcal{R} \to \mathcal{R}$, is a continuous solution of (1), if and only if either $f = 0$ or $f = 1$, or f has one of the following forms:

$$f(x) = |x|^c, x \in \mathcal{R}. \quad (21)$$

$$f(x) = |x|^c(signx), x \in \mathcal{R}. \quad (22)$$

With a certain $c \in \mathcal{R}$, and $c > 0$.

Proof. First, let's show that $f(x) > 0$ for all $x \neq 0$. From the defining equation(1), we see that$f(x) = \left(f(\sqrt{x})\right)^2$. This implies that $f(x) \geq 0, \forall x \in R+$. Now, if there exists a point $x_0 \neq 0$, such that $f(x_0) = 0$, then,

$$f(x) = f(\frac{x}{x_0} \cdot x_0) = f(\frac{x}{x_0}) f(x_0) = 0,$$

for all $x \in R$. Therefore, the function f would vanish identically. Since we are looking for non-vanishing functions f, there cannot be any x_0 with the above property. In otherworld's, $f(x) > 0, \forall x \in R$.
Now, we define,

$$g(x) = ln\, f(x). \quad (23)$$

Then the definition of relation (1), by taking the logarithm of the two sides can be write in terms of the function g as:

$$g(x) + g(y) = g(x\, y). \quad (24)$$

Again let,$h(x) = g(e^x)$, $\Leftrightarrow$ $g(x) = h(\ln x)$. By substitute in (24), we obtain:

$$h(\ln x) + h(\ln y) = h(\ln x + \ln y) \quad (25)$$

This relation by(20) has the solution $h(lnx) = c\, lnx$ and therefore

$$g(x) = c \ln x \quad (26)$$

From (23) and(26), we obtain:

$$ln\, f(x) = c \ln x = ln|x|^c.$$

Therefore, we get: $f(x) = |x|^c. x \in \mathcal{R}$.
Assume that, f is a constant function, say $f = c$ then from (1) we have:
$c = c^2$, which implies that $c = 0$ or $c = 1$, hence,$f(x) = 0$ or $f(x) = 1$.
Now, the proof of the theorem is complete.

Theorem 3.7. The general real valued solution, not identically 0, of $f(xy) = f(x)f(y)$is continuous at point, such that $\det(x) \neq 0$ is given by one of the following:

$$f(x) = |detx|^c . \tag{27}$$

$$f(x) = |detx|^c sign(detx). \tag{28}$$

Proof: By theorem (3.4) $f = 0, f = 1$ or f has the form (18), (19)where $g: \mathcal{R} \to \mathcal{R}$, is an additive function , we have got $g(t) = \log(f(e^t))$, since f is continuous so is also g , and consequently $g(t) = ct$, with certain $c \in \mathcal{R}$. Then we have: $ct = \log(f(e^t))$, Then, $e^{ct} = f(e^t)$, implies that : $(e^t)^c = f(e^t)$, let $x = \det(x) = e^t$, implies that $t = \ln|\det(x)|$, then $c \ln|\det(x)| = \log(f(\det(x)))$

$$f(\det(x)) = e^{\ln|\det(x)|^c}$$

Then, we get $f(x) = |detx|^c$.
This yields to the following important theorem:

4. Main results.

Assume we have $g(t) = \ln f(e^t)$, and since f is continuous and additive , we get $\ln f(e^t) = \alpha t, \Rightarrow f(e^t) = e^{\alpha t}$, interchange (e^t) with $x + y$,then $f(x+y) = (x+y)^\alpha$,similarly $f(x-y) = (x-y)^\beta$ .then from (3) and since f is continuous , so M_1, M_2. This yields to our first main result:

Theorem 4.1. The continuous general solution $f: \mathcal{R}^2 \to \mathcal{R}$ of the functional equation (2) given by:

$$f(x,y) = (x+y)^\alpha (x-y)^\beta. \tag{29}$$

For all $x, y \in \mathcal{R}$, where α and β are arbitrary real constant such that the domain of f is $\mathcal{R}^2$.
Proof:
Step (I): suppose that f is identically a constant, say $f \equiv c$ .then from(2), we have $c^2 = c$ which implies c = zero or c = 1.Hence the identically constant solutions of (2) are $f(x,y) = zero$ and$f(x,y) = 1$.
Step (II): Assume that f is not identically constant or continuous, that is $f \neq c$ then let us define a function, $f: \mathcal{R}^2 \to \mathcal{R}$ by:

$$F(x,y) = f(\tfrac{x+y}{2}, \tfrac{x-y}{2}) \tag{30}$$

for all $x, y \in \mathcal{R}$,using 30 in the equation (2) we get the following:
$F[(x+y)(u+v), (x-y)(u-v)] =$

$$F(x+y, x-y) \cdot F(u+v, u-v) \tag{31}$$

For all $x, y, u, v \in \mathcal{R}$, now let: $x + y = x_1,\ x - y = y_1, u + v = x_2,\ x - y = y_2,$ and by substituting in (31), we have

$$F(x_1x_2, y_1y_2) = F(x_1, y_1) \cdot F(x_2, y_2) \tag{32}$$

for all $x_1, x_2, y_1, y_2 \in \mathcal{R}$. Setting, $y_1 = y_2 = 1$, we get

$$F(x_1x_2, 1) = F(x_1, 1) \cdot F(x_2, 1) \tag{33}$$

for all $x_1, x_2 \in \mathcal{R}$ .But $F(x_1, 1)$ and $F(x_2, 1)$ are defined multiplicative functions as follows:

$$M_1(x_1) = F(x_1, 1) \tag{34}$$

$$M_1(x_2) = F(x_2, 1) \tag{35}$$

for all $x_1, x_2 \in \mathcal{R}$, then(33) reduces to:

$$M_1(x_1x_2) = M_1(x_1) \cdot M_1(x_2) \tag{36}$$

for all $x_1, x_2 \in \mathcal{R}$.Hence $M_1: \mathcal{R} \rightarrow \mathcal{R}$ is a multiplicative function.

Similarly, Setting, $x_1 = x_2 = 1$, in (32) we get

$$F(1, y_1 y_2) = F(1, y_1) \cdot F(1, y_2) \tag{37}$$

for all $y_1, y_2 \in \mathcal{R}$, defining $M_2: \mathcal{R} \rightarrow \mathcal{R}$ by

$$M_2(y) = F(1, y) \tag{38}$$

for all $x_1, x_2 \in \mathcal{R}$, then(37) reduces to:

$$M_1(y_1 y_2) = M_1(y_1) \cdot M_1(y_2) \tag{39}$$

for all $y_1, y_2 \in \mathcal{R}$ and hence $M_2: \mathcal{R} \rightarrow \mathcal{R}$ is a multiplicative function.

Now letting, $y_1 = x_2 = 1$ in equation(32), we obtain:

$$F(x_1, y_2) = F(x_1, 1) \cdot F(1, y_2) \tag{40}$$

for all $x_1, y_2 \in \mathcal{R}$,which yields

$$F(x_1 y_2) = M_1(x_1) \cdot M_2(y_2) \tag{41}$$

for all $x_1, y_2 \in \mathcal{R}$.

Now using (41) in (30), we have

$$f(x, y) = F(x + y, x - y) = M_1(x + y) \cdot M_2(x - y)$$

for all $x, y \in \mathcal{R}$.which is a solution of (2) .

Step (III): Assume that f identically not constant but continuous, then f has the following form: see(18)

$$f(x) = \exp[g(ln|x|)], x \neq 0$$

Where g is an additive function, here we get

$$g(t) = \ln f(e^t)$$

And so f is continuous, implies that $g(t)$ is also continuous, then

$$g(t) = at$$

So,

$$\ln f(e^t) = at$$
$$f(e^t) = e^{at},$$

Instead of e^t put $x + y$.
We get,

$$f(x + y) = (x + y)^a \tag{42}$$

Similarly,

$$f(x - y) = (x - y)^b. \tag{43}$$

Then,

$$f(x, y) = f(x + y) \cdot f(x - y)$$
$$f(x, y) = (x + y)^a (x - y)^b.$$

For all $x_1, y_2 \in \mathcal{R}$.

Therefore, the proof of the theorem is now complete.
Theorem 4.2. The continuous general solution $f: \mathcal{R}^2 \to \mathcal{R}$ of the functional equation (6) given by:

$$f(x,y) = |x^2 - y^2|^\alpha - 1. \quad (44)$$

For all $x, y \in \mathcal{R}$, where α is arbitrary real constant.
Proof:
Step (I): suppose that f is identically a constant, say $f \equiv k$ .then from (6), we have $k^2 + 2k = k$ which implies $k = zero$ or $k = -1$.Hence the identically constant solutions of (6) are $f(x,y) = zero$ and $f(x,y) = -1$.
Step (II): Assume that f is neither constant nor continuous, that is $f \neq c$ then let us define a function, $f: \mathcal{R}^2 \to \mathcal{R}$ by:

$$F(x,y) = f\left(\frac{x+y}{2}, \frac{x-y}{2}\right) + 1 \quad (45)$$

for all $x, y \in \mathcal{R}$,using (45) in the equation (6) we get the following:
$F[(x+y)(u+v), (x-y)(u-v)] =$

$$F(x+y, x-y) \cdot F(u+v, u-v) \quad (46)$$

For all $x, y, u, v \in \mathcal{R}$, now let: $x + y = x_1,\ x - y = y_1, u + v = x_2,\ x - y = y_2,$ and by substituting in (46), we have

$$F(x_1x_2, y_1y_2) = F(x_1, y_1) \cdot F(x_2, y_2) \quad (47)$$

for all $x_1, x_2, y_1, y_2 \in \mathcal{R}$.

Case (1): Setting, $y_1 = x_2 = 1$, we get

$$F(x_1y_2, 1) = F(x_1, 1) \cdot F(1, y_2) \quad (48)$$

$$F(y_2x_1, 1) = F(y_2, 1) \cdot F(1, x_1) \quad (49)$$

Implies that,

$$F(x_1, 1) \cdot F(1, y_2) = F(y_2, 1) \cdot F(1, x_1) \quad (50)$$

$$F(1, y_2) = \frac{F(1,x_1)}{F(x_1,1)} \cdot F(y_2, 1) \quad (51)$$

$\frac{F(1,x_1)}{F(x_1,1)} = constant$, say $\alpha \neq 0$, then:

$$F(1, y_2) = \alpha\, F(y_2, 1) \quad (52)$$

Then letting $x_1 = x_2 = 1$ (47), and using(52), we obtain

$$F(y_1y_2, 1) = \alpha\, F(y_1, 1) \cdot F(y_2, 1) \quad (53)$$

for all $x_1, y_2 \in \mathcal{R}$. Now defining $M: \mathcal{R} \to \mathcal{R}$ by:

$$M(x) = \alpha F(x, 1) \quad (54)$$

for all $x \in \mathcal{R}$, then(53) reduces to:

$$M(y_1y_2) = M(y_1) \cdot M(y_2) \quad (55)$$

for all $y_1, y_2 \in \mathcal{R}$.Hence $M: \mathcal{R} \to \mathcal{R}$ is a multiplicative function.

Similarly, Setting, $y_1 = y_2 = 1$, in (47) we get

$$F(x_1x_2, 1) = F(x_1, 1) \cdot F(x_2, 1) \tag{56}$$

for all $x_1, x_2 \in \mathcal{R}$, which by (54) yields

$$F(x_1x_2) = k\, M(x_1) \cdot M(x_2) \tag{57}$$

for all $x_1, x_2 \in \mathcal{R}$, where $k = \frac{1}{\alpha^2}$, which implies that $k = 1$, from (47).

by using (45),we have

$$f\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = F(x, y) - 1$$

Now using(57), we have:

$$f(x, y) = F(x+y, x-y) - 1$$

$$= M(x+y) \cdot M(x-y) - 1$$

$$= M(x^2 - y^2) - 1$$

For all $x, y \in \mathcal{R}$.

Step (III): Assume that f identically not constant but continuous, then f has the following form: see(18)

$$f(x) = \exp[g(ln|x|)], x \neq 0$$

Where g is an additive function,

But, $\ln(f(x) + 1) = \ln f(x)$, since f is continuous and satisfied Cauchy equation. Then,

$$\ln(f(x) + 1) = g[\ln(x)] \tag{58}$$

Assume that, $x = e^t$, then we have:

$$\ln(f(e^t) + 1) = g[\ln(e^t)]$$

$$\ln(f(e^t) + 1) = g(t)$$

$$\ln(f(e^t) + 1) = ct. \tag{59}$$

$$f(e^t) + 1 = e^{ct}$$

Again, let $e^t = detx$

We get,

$$f(detx) + 1 = (detx)^c, \text{ but } detx = x^2 - y^2$$

$$f(x, y) = |x^2 - y^2|^c - 1.$$

For all $x, y \in \mathcal{R}$, where α is arbitrary real constant

Theorem 4.3. The continuous general solution $f: \mathcal{R}^2 \to \mathcal{R}$ of the functional equation (4) given by:

$$f(x, y) = |x^2 - y^2|^\beta. \tag{60}$$

For all $x, y \in \mathcal{R}$, where α is arbitrary real constant.

Proof: Step (I) and (II) same as the above theorem(4.2).

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Step (III): Assume that f identically not constant but continuous, then f has the following form: see(18)

$$f(x) = \exp[g(ln|x|)], x \neq 0$$

Where g is an additive function,

$$\ln\big(f(x)\big) = g[\ln(x)]$$

Assume that, $x = e^t$, then we have:

$$\ln\big(f(e^t)\big) = g[\ln(e^t)]$$

$$\ln\big(f(e^t)\big) = g(t)$$
$$\ln\big(f(e^t)\big) = \beta t$$
$$f(e^t) = e^{\beta t}$$

Again, let $e^t = |detx|$,
Then, we get,

$$f(detx) = |detx|^{\beta}, \text{ but } detx = x^2 - y^2$$

$$f(x,y) = |x^2 - y^2|^{\beta}.$$

For all $x, y \in \mathcal{R}$, where β is arbitrary real constant.
Theorem 4.4. The continuous general solution $f: \mathcal{R}^2 \to \mathcal{R}$ of functional equation (10) given by:

$$f(x,y) = |x^2 + y^2|^{\alpha} - 1. \qquad (61)$$

For all $x, y \in \mathcal{R}$, where α is arbitrary real constant.
Proof: Step (I) and (II) are investigated before, see[7].
Step (III): Assume that f identically not constant but continuous, then f has the following form: see(18)

$$f(x) = \exp[g(ln|x|)], x \neq 0$$

Where g is an additive function, But, $\ln(f(x)+1) = \ln f(x)$, since f is continuous and satisfied Cauchy equation.

$$\ln(f(x)+1) = g[\ln(x)]$$

Assume that, $x = e^t$, then we have:

$$\ln(f(e^t)+1) = g[\ln(e^t)]$$

$$\ln(f(e^t)+1) = g(t)$$
$$\ln(f(e^t)+1) = \alpha t$$
$$f(e^t) + 1 = e^{\alpha t}$$

Again, let $e^t = |per\ x|$.

We get,

$$f(per\ x) = |per\ x|^{\alpha} - 1, \text{ but } per\ x = x^2 + y^2$$

$$f(x,y) = |x^2 + y^2|^{\alpha} - 1.$$

For all $x, y \in \mathcal{R}$, where α is arbitrary real constant.
Application: (General form of rectangle area): The area $F(x,y)$ of a rectangle with sides x and y, is the general solution of the following equations:

Table 2: rectangle diagram

y	x_1	x_2	x_3	…	x_n	

$$F(x, y_1 + y_2 + \cdots + y_n) = F(x, y_1) + F(x, y_2) + \cdots + F(x, y_n) \qquad (62)$$

$F(x_1+x_2+\cdots+x_n,y)=F(x_1,y)+F(x_2,y)+\cdots+F(x_n,y)$ (63) $x,y\in\mathfrak{R}_+^*$, with regularity condition that $F>0$ and continuous in the nth variables, given by:

$$F(x,y)=c_0xy, \text{ with } F(1,1)=1 \quad (64)$$

Then, the area is:

$$F(x,y)=xy. \quad (65)$$

Proof: by using theorem (3.5), such that >0 , and continuous we have the following form:

$$F(x,y)=cxy \quad (66)$$

Plugging equation (66) in equation (62), we get:

$$cx(y_1+y_2+y_3+\cdots+y_n)=c_1xy_1+c_2xy_2+c_3xy_3+\cdots+c_nxy_n$$

Setting x =1, we obtain:

$$c(y_1+y_2+y_3+\cdots+y_n)=cy_1+cy_2+cy_3+\cdots+cy_n$$

Which extend Cauchy theorem again, and since F is continuous in the variable y, it follows that c is a continuous function, here we have:

$\lim_{i\to n} y_i=y$, so

$$c(y) = c_0y, \text{ for all } y\in R \quad (67)$$

From this, we conclude that: $F(x,y)=c_0xy$, with $F(1,1)=c_0$,for all $x,y\in R$.

Then the area is: $F(x,y)=xy$

Corollary 4.6. The continuous general solution $f:\mathcal{R}^2\to\mathcal{R}$ of the functional equation (8) given by:

$$f(x,y)= |x^2+y^2|^{\alpha}. \quad (68)$$

For all $x,y\in\mathcal{R}$, where α is arbitrary real constant.

Corollary 4.7. The continuous general solution $f:\mathcal{R}^2\to\mathcal{R}$ of the functional equation (13) given by:

$$f(x,y)= |x^2+y^2|^{c}V(\theta_{xy}). \quad (69)$$

For all $x,y\in\mathcal{R}$, where c is arbitrary real constant.

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"الحل العام المستمر للمعادلة الدالية المتعلقة بخواص المصفوفة المتماثلة"

ملخص البحث:

في هذه الورقة العلمية، تم إيجاد الحل العام المستمر واثباته لكل من المعادلات الدالية التالية:

1. $f(\mathrm{ux}+\mathrm{vy},\mathrm{uy}+\mathrm{vx}) = f(x,y)f(u,v), x,y,u,v\in R.$
2. $f(\mathrm{ux}-\mathrm{vy},\mathrm{uy}-\mathrm{vx}) = f(x,y)f(u,v), x,y,u,v\in R.$
3. $f(\mathrm{ux}-\mathrm{vy},\mathrm{uy}-\mathrm{vx}) = f(x,y)f(u,v)+f(x,y)+f(u,v), x,y,u,v\in R.$
4. $f(\mathrm{ux}+\mathrm{vy},\mathrm{uy}-\mathrm{vx}) = f(x,y)f(u,v)+f(x,y)+f(u,v), x,y,u,v\in R.$

والتي تولدت من خواص المصفوفات المتماثلة من النوع 2×2، وهذه النتائج يمكن تطبيقها على عدد من المعادلات الدالية ذات الصلة، و يكون الحل العام المستمر دالة كثيرة حدود، في حالة الثابت الاختياري عدداً طبيعياً.