العدد الرابع والعشرون
تاريخ الإصدار: 2 – تشرين الأول – 2020 م
ISSN: 2663-5798
www.ajsp.net

"Calculus of Variations on Fiber Bundles"

Researcher:

Tahani Hassan Ahmed Alballa

Sudan University of Science and Technology

Abstract

In this paper we introduce the variational bicomplex for a fibered manifold, sections of a bundle, and we develop the requisite calculus of vector fields and differential forms on the infinite jet bundle of such spaces. We pay particular attention to the notion of generalized vector fields.

Keywords: Manifolds, Sections, Jet bundles, Vector fields, variational bicomplex.

1. Introduction

The variational bicomplex is a double complex of differential forms defined on the infinite jet bundle of any fibered manifold $\pi : E \to M$. This double complex of forms is called the variational bicomplex because one of its differentials (or, more precisely, one of the induced differentials in the first term of the first spectral sequence) coincides with the classical Euler-Lagrange operator, or variational derivative, for arbitrary order, multiple integral problems in the calculus of variations.Thus, the most immediate application of the variational bicomplex is that of providing a simple, natural, and yet general, differential geometric development of the variational calculus. Indeed, the subject originated within the last fifteen years in the independent efforts of W. M. Tulczyjew and A. M. Vinogradov [1] to resolve the Euler-Lagrange operator and thereby characterize the kernel and the image of the the variational derivative. But the utility of this bicomplex extents well beyond the domain of the calculus of variations. Indeed, it may well be that the more important aspects of our subject are those aspects which pertain either to the general theory of conservation laws for differential equations, as introduced by Vinogradov, or to the theory of characteristic (and secondary characteristic) classes and Gelfand-Fuks cohomology [1], as suggested by T. Tsujishita. All of these topics are part of what I. M. Gelfand [1], in his 1970 address to the International Congress in nice, called formal differential geometry. The variational bicomplex plays the same ubiquitous role in formal differential geometry, that is, in the geometry of the infinite jet bundle for the triple (E, M, π) that the de Rham complex [1] plays in the geometry of a single manifold M .

2. Sections of fiber bundles

(2.1)Definition

Consider a generic fiber bundle $\pi : E \to M$ with generic fiber F . We name section of the bundle a rule s that to each point $p \in M$ of the based manifold associates a point $s(p) \in F_p$ in the fiber above p , namely a map

$$s : M \mapsto E ,$$

such that:

$$\forall p \in M : s(p) \in \pi^{-1}(p) .$$

The above definition is illustrated in **Fig. 1** which also clarifies the intuitive idea standing behind the chosen name for such a concept.

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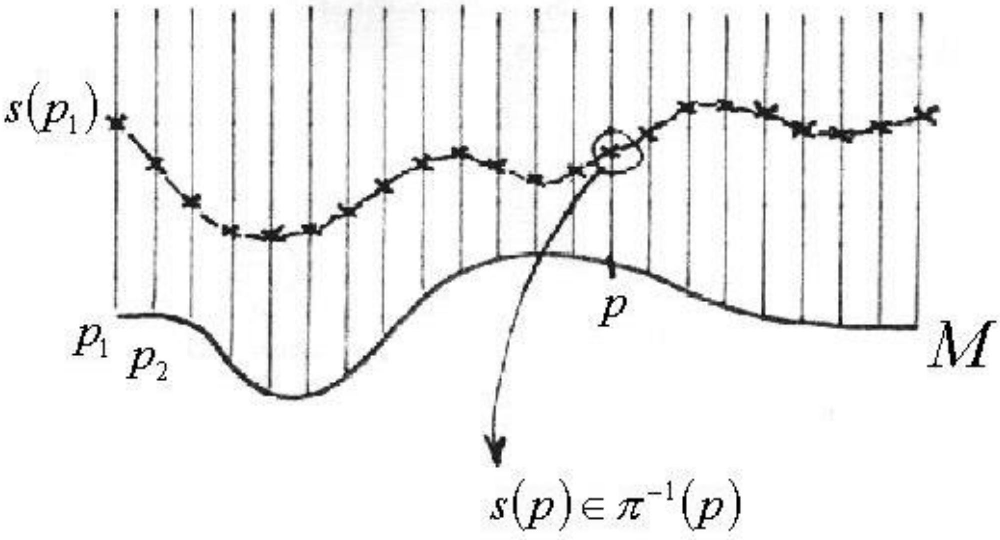

Figure 1: A section of a fiber bundle [3] Frè, P.G.

It is clear that sections of the bundle can be chosen to be continuous, differentiable, smooth or, in the case of complex manifolds, even holomorphic, depending on the properties of the map S in each local trivialization of the bundle. Indeed given a local trivialization and given open charts for both the base manifold M and for the fiber F the local description of the section reduces to a map:

$$U \subset R^m \mapsto F_U \subset R^n$$

where m and n are the dimensions of the base manifold and of the fiber respectively.

Example (2.2) holomorphic vector fields on S^2

As we have seen above the 2-sphere S^2 is a complex manifold of complex dimension one covered by an atlas composed by two charts, that of the North Pole and that of the South Pole (see Fig. **2**)

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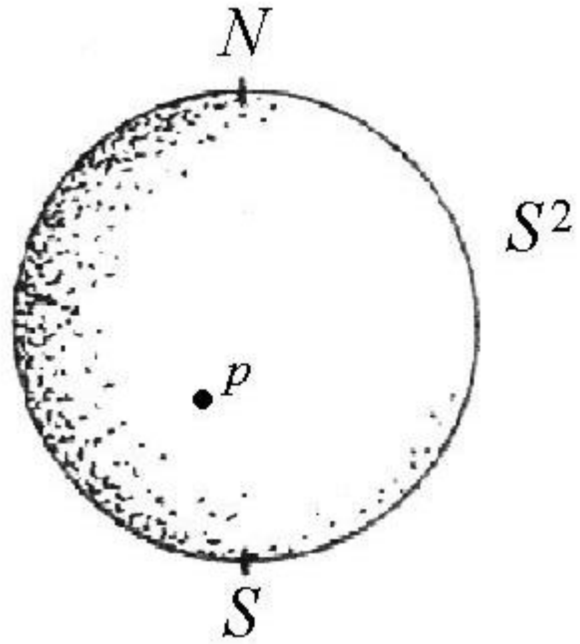

Figure 2: A section of a fiber bundle [3] Frè, P.G.

We are specifically interested in smooth sections, namely in section that are infinitely differentiable. Given a bundle $\pi : E \rightarrow M$, the set of all such sections is denoted by:

$$\Gamma(E,M).$$

Of particular relevance are the smooth sections of vector bundles. In this case to each point of the base manifold p we associate a vector $\vec{\upsilon}(p)$ in the vector space above the point p . In particular we can consider sections of the tangent bundle TM associated with a smooth manifold M . Such sections correspond to the notion of vector fields.

Definition (2.3)

Given a smooth manifold M , we name vector field on M a smooth section $\vec{t} \in \Gamma(TM, M)$ of the tangent bundle. The local expression of such vector field in any open chart (U, φ) is

$$\vec{t} = t^{\mu}(x) \frac{\vec{\partial}}{\partial x^{\mu}} \quad \forall x \in U \subset M .$$

3. Infinite Order Jet Bundles

Let $\pi : E \to M$ be a smooth fibered manifold with total space E of dimension $n+m$ and base space M of dimension n. The projection map π is a smooth surjective submersion. The fiber $\pi^{-1}(x)$ over a point $x \in M$ may change topologically as x varies over M; for example, let E be $R^2 - \{(1,0)\}$ and let π be the projection onto the x axis. In many situations E will actually be a fiber bundle over M but this additional structure is not needed to define the variational bicomplex. We assume that M is connected.

We refer to the fibered manifold E locally by coordinate charts (φ, U) where, for $p \in U \subset E$,

$$\varphi(p) = (x(p), u(p))$$

and

$$x(p) = (x^i(p)) = (x^1, x^2, ..., x^n), \qquad u(p) = (u^\alpha(p)) = (u^1, u^2, ..., u^m).$$

These coordinates are always taken to be adapted to the fib- ration π in the sense that (φ_0, U_0), where $\varphi_0 = \varphi \circ \pi$ and $U_0 = \pi(U)$, is a chart on the base manifold M and that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & R^n \times R^m \\ \pi \downarrow & & \downarrow proj \\ U_0 & \xrightarrow{\varphi_0} & R^n \end{array},$$

where $proj((x,u)) = (x)$, commutes. If (ψ, V) is an overlapping coordinate system and $\psi(p) = (y(p), \upsilon(p))$ then on the overlap $U \cap V$ we have the change of coordinates formula

$$y^j = y^j(x^i) \text{ and } \upsilon^\beta = \upsilon^\beta(x^i, u^\alpha). \qquad \textbf{(3.1)}$$

If $\rho : F \to N$ is another fibered manifold, then a map $\phi : E \to F$ is said to be fiber-preserving if it is covers a map $\phi_0 : M \to N$, i.e., the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ \pi \downarrow & & \downarrow \rho \\ M & \xrightarrow{\phi_0} & N \end{array}$$

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commutes. Thus, the fiber over $x \in M$ in E is mapped by ϕ into the fiber over $y = \phi_0(x) \in N$ in F. We shall, on occasion, consider arbitrary maps between fibered bundles although the general theory of the variational bicomplex is to be developed within the category (which is defined following) of fibered manifolds and fiber-preserving maps.

(3.2) Definition

Let M be a compact, n dimensional C^∞ manifold, possibly with boundary. Denote by $FB(M)$ the category of C^∞ fiber bundles over M. A morphism $f : E \to F$ of $FB(M)$ is a C^∞ map such that $f_x = f|E_x$ maps E_x into F_x for each $x \in M$, where E_x is a fiber of E at x. If E and F are C^∞ vector bundles and each f_x is linear we call f a vector bundle morphism. We denote by $VB(M)$ the category of C^∞ vector bundles over M and vector bundle morphisms and by $FVB(M)$ the mongrel category of C^∞ vector bundles over M and fiber bundle morphisms.

(3.3) Definition

Suppose that there is a given strictly positive measure μ on M and we let E denote a C^∞ fiber bundle over M. The space of k -jets of local sections of E, regarded as a C^∞ fiber bundle over E will be denoted by $J_0^k(E)$. It can also be regarded as a C^∞ fiber bundle in which case we denote it by $J^k(E)$. As usual $j_k : C^\infty(E) \to C^\infty(J^k(E))$ denotes the k -jets extension map.

If F is a C^∞ real valued function on $J^k(E)$ then for each $s \in C^\infty(E)$ we get a real valued function $L(s)$ on M by $L(s)(x) = F\left(j_k(s)_x\right)$. Such a function $L : C^\infty(E) \to C^\infty(R_M)$ (where $R_M = M \times R$ is the product line bundle over M) is called a k th order Lagrangian for E. The set of all k th order Lagrangians for E is a vector space $\mathrm{Lgn}_k(E)$.

Denote by $\pi^k : J^k(E) \to M$ the fiber bundle of k -jets of local sections of E.

The fiber $(\pi^k)^{-1}(x)$ of $x \in M$ in $J^k(E)$ consists of equivalence classes, denoted by $j^k(s)(x)$, of local sections s of E at x ; two local sections s_1 and s_2 about x are equivalent if with respect to some adapted coordinate chart (and hence any adapted chart) all the partial derivatives of s_1 and s_2 agree up to order k at x.

Each projection $\pi_k^l : J^l(E) \to J^k(E)$, defined for $l \ge k$ by

$$\pi_k^l\left[j^l\left((s)(x)\right)\right] = j^k(s)(x),$$

is a smooth surjection and, in fact, for $l = k+1$ defines $J^l(E)$ as an affine bundle over $J^k(E)$.This implies that for all $l \ge k$, $J^l(E)$ is smoothly contractible to $J^k(E)$.

We shall often write, simply for the sake of notational clarity,

$$\pi_E^k = \pi_0^k \text{ and } \pi_M^k = \pi^k .$$

for the projections from $J^k(E)$ to E and M .

An adapted coordinate chart (φ, U) on E lifts to a coordinate chart $(\tilde{\varphi}, \tilde{U})$ on $J^k(E)$. Here $\tilde{U} = (\pi_E^k)^{-1}(U)$ and, if $s : U_0 \to U$ is the section $s(x) = (x^i, s^\alpha(x^i))$, then the coordinates of the point $j^k(s)(x)$ are

$$\textbf{(3.4)}\quad \tilde{\varphi}[j^k(s)(x)] = (x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, \ldots, u^\alpha_{i_1 i_2 \ldots i_k}),$$

where, for $l = 0,1,\ldots,k$,

$$u^\alpha_{i_1 i_2 \ldots i_l} = \frac{\partial^l s^\alpha}{\partial x^{i_1} \partial x^{i_2} \ldots \partial x^{i_l}}(x),$$

and where $1 \le i_1 \le i_2 \le \ldots \le i_l \le n$.

The inverse sequence of topological spaces $\{J^k(E), \pi_k^l\}$ determine an inverse limit space $J^\infty(E)$ together with projection maps

$$\pi_k^\infty : J^\infty(E) \to J^k(E) \text{ and } \pi_E^\infty : J^\infty(E) \to E,$$

and

$$\pi_M^\infty : J^\infty(E) \to M .$$

The topological space $J^\infty(E)$ is called the infinite jet bundle of the fibered manifold E . A point in $J^\infty(E)$ can be identified with an equivalence class of local sections around a point $x \in M$ — local sections s around x define the same point $j^\infty(s)(x)$ in $J^\infty(E)$ if they have the same Taylor coefficients to all orders at x . A basis for the inverse limit topology on $J^\infty(E)$ consists of all sets $\tilde{W} = (\pi_k^\infty)^{-1}(W)$, where W is any open set in $J^k(E)$ and $k = 0,1,2,\ldots$.

If σ is a point in $J^\infty(E)$ it will be convenient to write

$$\sigma^k = \pi_k^\infty(\sigma)$$

for its projection into $J^k(E)$.

The notion of a smooth function on the infinite jet bundle must be defined.

(3.5) Definition

Let p be any manifold and let $C^\infty(J^k(E), P)$ be the set of smooth maps from $J^k(E)$ to P. The set of smooth functions from $J^\infty(E)$ to P is denoted by $C^\infty(J^\infty(E), P)$. If $f \in C^\infty(J^\infty(E), P)$ then, f must factor through a smooth map $\hat{f}$ from $J^k(E)$ to P for some k , i.e.,

$$\textbf{(3.6)}\quad f = \hat{f} \circ \pi_k^\infty$$

We call k the order of f. If f is of order k, then it also of any order greater than k. In particular, the projections maps π_k^∞ are themselves smooth functions of order k.

We let $C^\infty(J^\infty(E))$ denote the set of smooth, real-valued functions on $J^\infty(E)$. If f is a smooth, real-valued function on $J^\infty(E)$ which is represented by a smooth function $\hat{f}$ on $J^k(E)$, then on each coordinate neighborhood $\left(\pi_E^\infty\right)^{-1}(U)$ and for each point $\sigma = j^\infty(s)(x) \in \left(\pi_E^\infty\right)^{-1}(U)$ with k -jet coordinates given by **(3.4)**,

$$\textbf{(3.7)}\quad f(\sigma) = \hat{f}\left(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, \ldots, u^\alpha_{i_1 i_2 \ldots i_k}\right).$$

A map $f : P \to J^\infty(E)$ is said to be smooth if for any manifold Q and any smooth map $g : J^\infty(E) \to Q$, the composition $g \circ f$ from P to Q is a smooth map. Likewise, if $\rho : F \to N$ is another fibered manifold we declare that a map $\Phi : J^\infty(E) \to J^\infty(F)$ is smooth if for every smooth map $g : J^\infty(F) \to Q$ the composition $g \circ \Phi$ from $J^\infty(E)$ to Q is smooth.

A map $\Phi : J^\infty(E) \to J^\infty(F)$ called projectable if it is covers maps from $J^k(E)$ to $J^k(F)$ for each k, i.e.,

$$\begin{array}{ccc} J^\infty(E) & \xrightarrow{\Phi} & J^\infty(F) \\ \pi_k^\infty \downarrow & & \downarrow \rho_k^\infty \\ J^k(E) & \xrightarrow{\Phi_k} & J^k(F) \end{array}.$$

Such a map is of type $(0,1,2,\ldots)$.

Although the fibered manifold $\pi : E \to M$ may not admit any global sections, the bundle $\pi_E^\infty : J^\infty(E) \to E$ always admits global sections. These can be readily constructed using partitions of unity.

An important class of smooth maps from $J^\infty(E)$ to $J^\infty(F)$ are those which arise as the prolongation of maps from E to F.

(3.8) Definition

Let ϕ be a map from E to F which covers a local diffeomorphism ϕ_0. Then the infinite prolongation of ϕ is the map

$$pr\phi : J^\infty(E) \to J^\infty(F)$$

defined by

$$\textbf{(3.9)}\quad pr\phi(j^\infty(s)(x)) = \left[j^\infty\left(\phi \circ s \circ \phi_0^{-1}\right)\right]\!\left(\phi_0(x)\right),$$

where s is a local section of E defined on a neighborhood of x on which ϕ_0 is a diffeomorphism.

ISSN: 2663-5798

The prolongation of ϕ is a smooth, projectable map. Moreover, if ϕ is a diffeomorphism, then so is prϕ.

4. Vector Fields and Generalized Vector Fields

The tangent bundle to the infinite jet bundle $J^{\infty}(E)$ can be defined in various (equivalent) ways. One possibility is to consider the inverse system of tangent bundles $T(J^{k}(E))$ with the projections $(\pi_k^l)^*$ from $T(J^{l}(E))$ to $T(J^{k}(E))$ for all $l \geq k$ as connecting maps and to designate $T(J^{\infty}(E))$ as the inverse limit of these vector bundles. In this way $T(J^{\infty}(E))$ inherits the structure of a topological vector bundle over $J^{\infty}(E)$ .Alternatively, the tangent space $T_{\sigma}(J^{\infty}(E))$ at a point $\sigma \in J^{\infty}(E)$ may be defined directly as the vector space of real-valued R linear derivations on $J^{\infty}(E)$ .The tangent bundle $T(J^{\infty}(E))$ can then be constructed from the union of all individual

tangent spaces $T_{\sigma}(J^{\infty}(E))$ in the usual fashion. These two approaches are equivalent. Indeed, a derivation X_{σ} on $J^{\infty}(E)$ at the point σ determines a sequence of derivations X_{k,σ^k} to $T(J^{k}(E))$ at $\sigma^k = \pi_k^{\infty}(\sigma)$ — if f is a smooth function on $J^{k}(E)$, then

$$X_{k,\sigma^k}(f) = X_{\sigma}(f \circ \pi_k^{\infty}) \quad \textbf{(4.1)}$$

These derivations satisfy

$$(\pi_k^l)^* X_{l,\sigma^l} = X_{k,\sigma^k} \quad \textbf{(4.2)}$$

for all $l \geq k$ and therefore define a tangent vector in the inverse limit space $T(J^{\infty}(E))$ at σ. Conversely, every sequence of vectors $X_{k,\sigma^k} \in T_{\sigma^k}(J^{\infty}(E))$ satisfying **(4.2)** defines a derivation X_p on $J^{\infty}(E)$ at σ — if f is a function on $J^{\infty}(E)$ which is represented by a function $\hat{f}$ on $J^{k}(E)$, then

$$X_{\sigma}(f) = X_{k,\sigma^k}(\hat{f}).$$

The projection property **(4.2)** ensures that this is a well-defined derivation, independent of the choice of representative $\hat{f}$ of f .

If X_p is represented by the sequence of vectors X_k at σ^k for $k = 0,1,2,...$ and Φ is represented, by functions Φ_k^{mk} then $\Phi^*(X_p)$ is represented by the sequence of vectors $(\Phi_k^{mk})^*(X_{mk})$.

A vector field X on $J^{\infty}(E)$ is defined to be a $C^{\infty}(J^{\infty}(E)$ valued, R-linear derivation on $C^{\infty}(J^{\infty}(E)$ .Thus, for any real-valued function f on $J^{\infty}(E)$, $X(f)$ is a smooth function on $J^{\infty}(E)$ and must therefore be of some finite order. Although the order of the function $X(f)$ may exceed that of f , the order of $X(f)$ is nevertheless bounded for all functions f of a given order.

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ISSN: 2663-5798 www.ajsp.net

Proposition (4.3)

Let X be a vector field on $J^{\infty}(E)$ Then for each $k=0,1,2,...$, there exists an integer m_k such that for all functions f of order k , the order of $X(f)$ does not exceed m_k .

Proof:

The case k = 0 and E compact is easily treated. For k = 0 and E non compact or for k > 0, we argue by contradiction. First, pick a sequence of points p_i , $i=1,2,3,...$ in $J^k(E)$ with no accumulation points. Let U_i be a collection of disjoint open sets in $J^k(E)$ containing p_i . Let ϕ_i be smooth functions on $J^k(E)$ which are 1 on a neighborhood of p_i and have support inside of U_i .

Now suppose, contrary to the conclusion of the proposition, that there are functions f_i on $J^k(E)$ for i = 1, 2, 3, . . . such that the order of $X(f_i)$ exceeds i . We can assume that the order of $X(f_i)$ exceeds i in a neighborhood of a point $\tilde{p}_i$, where $\tilde{p}_i \in \left(\pi_k^{\infty}\right)^{-1}(p_i)$. If this is not the case, if the maximum order of $X(f_i)$ is realized about a point $\tilde{q}_i \notin \left(\pi_k^{\infty}\right)^{-1}(p_i)$, then we can simply redefine f_i to be the composition

of f_i with any diffeomorphism of $J^k(E)$ which carries p_i to the point $q_i = \pi_k^{\infty}(\tilde{q}_i)$.

Define $f = \sum_i \phi_i f_i$. Then f is a smooth function on $J^k(E)$ but $X(f)$ is not a smooth function on $J^{\infty}(E)$ since it is not of global finite order. This contradiction proves the lemma.

We say that a vector field X on $J^{\infty}(E)$ is of type $(m_0, m_1, m_2, ...)$ if for all functions f of order k the order of $X(f)$ is m_k . With no loss in generality, we shall suppose the sequence m_k increases with k . A vector field on $J^{\infty}(E)$ is projectable if it projects under π_k^{∞} to a vector field on $J^k(E)$ for each k . Projectable vector fields are of type $(0,1,2,...)$.

With respect to our induced local coordinates on $J^{\infty}(U)$, a vector field X takes the form

$$X = a^i \frac{\partial}{\partial x^i} + b^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \sum_{p=1}^{\infty} \left[\sum_{1 \le i_1 \le i_2 \le .. \le i_p \le n} b^{\alpha}_{i_1 i_2 .. ip} \frac{\partial}{\partial u^{\alpha}_{i_1 i_2 .. ip}} \right]. \quad \textbf{(4.4)}$$

The components a^i, b^{α} and $b^{\alpha}_{i_1 i_2 .. i_p}$ are all smooth functions on $J^{\infty}(U)$ .If f is a smooth function on $J^{\infty}(U)$,then f is of finite order and so $X(f)$ involves only finitely many terms from **(4.4).** The vector field X is projectable if the a^i and b^{α} are smooth functions on U and the $b^{\alpha}_{i_1 i_2 .. i_k}$ are smooth functions on $J^k(U)$, $k=1,2,...$.

The sets of sections of $T(J^k(E))$ for $k = 0,1,2,...$ do not constitute an inverse system (since it is not possible to project an arbitrary vector field on $J^l(E)$ to one on $J^k(E)$ for $k < l$) and, for this reason, it is not possible to represent a given vector field on the infinite jet bundle by a sequence of vector fields on finite dimensional jet bundles. To circumvent this problem we introduce the notion of generalized vector fields. Generalized vector fields first appeared as generalized or higher order symmetries of the KdV equation. They play a central role in both the theory and applications of the variational bicomplex. First recall that if P and Q are finite dimensional manifolds and $\phi : P \to Q$ is a smooth map, then a vector field along ϕ is a smooth map $Z : P \to T(Q)$ such that for all $p \in P$, $Z(p)$ is a tangent vector to Q at the point $\phi(p)$.

Definition (4.5)

A generalized vector field Z on $J^k(E)$ is a vector field along the projection π_k^∞ , i.e., Z is a smooth map

$$Z : J^\infty(E) \to T(J^k(E))$$

such that for all $\sigma \in J^\infty(E)$, $Z(\sigma) \in T_{\sigma^k}(J^k(E))$.

Similarly, a generalized vector field Z on M is a vector field along the projection π_M^∞ , i.e., Z is a smooth map

$$Z : J^\infty(E) \to T(M)$$

such that for all $\sigma = j^\infty(s)(x)$, $Z(\sigma) \in T_x(M)$.

Since a generalized vector field Z on $J^k(E)$ is a smooth map from the infinite jet bundle to a finite dimensional manifold, it must factor through $J^m(E)$ for some $m \geq k$. Thus there is a vector field $\hat{Z}$ along π_k^m , i.e., a map

$$\hat{Z} : J^m(E) \to T(J^k(E))$$

such that

$$Z = \hat{Z} \circ \pi_m^\infty .$$

We call m the order of the generalized vector field Z . If f is a function on $J^k(E)$, then $Z(f)$ is the smooth function on $J^\infty(E)$ defined by

$$Z(f)(\sigma) = \hat{Z}(\sigma^m)(f) .$$

The order of the function $Z(f)$ is m . Note that a generalized vector field on $J^k(E)$ of order k is simply a vector field on $J^k(E)$.

Generalized vector fields are projectable. If Z is a generalized vector field on $J^l(E)$ then for $k \leq l$ the map $\overline{Z} : J^\infty(E) \to T(J^k(E))$ defined by

$\bar{Z}(\sigma) = (\pi_k^l)^*(\sigma^l)[(Z(\sigma))]$

is a generalized vector field on $J^k(E)$. We write $(\pi_k^l)^*(Z)$ for $\bar{Z}$.

Proposition (4.6)

Let X be a vector field on $J^\infty(E)$ of type $(m_0, m_1, m_2, ...)$. Then there exist generalized vector fields X_k on $J^k(E)$ of order m_k such that

$$(\pi_k^l)^*(X_l) = X_k \quad \textbf{(4.7)}$$

and, for all functions f of order k,

$$X(f)(\sigma) = X_k(\sigma)(f). \qquad \textbf{(4.8)}$$

Conversely, given a sequence of generalized vector fields X_k on $J^k(E)$ satisfying **(4.7))**, there exists a unique vector field X on $J^\infty(E)$ satisfying **(4.8)**.

Proof:

Given X, simply define the generalized vector fields X_k by

$$X_k(\sigma) = (\pi_k^\infty)^*(\sigma)(X_\sigma).$$

We remark that if the vector field X on $J^\infty(E)$ is given locally by **(4.4)**, then the associated generalized vector fields X_k on $J^k(E)$ are given by truncating the infinite sum on p in **(4.4)** at $p = k$.

5. Differential Forms

The p^{th} exterior product bundles $\Lambda^p(J^k(E))$ together with the pullback maps

$$(\pi_k^l)^* : \Lambda^p(J^k(E)) \to \Lambda^p(J^k(E)),$$

defined for all $l \geq k \geq 0$, form a direct system of vector bundles whose direct limit is designated as the p^{th} exterior product bundle $\Lambda^p(J^\infty(E))$ of $J^\infty(E)$. Let $\sigma \in J^\infty(E)$. Then each $\omega \in \Lambda_\sigma^p(J^\infty(E))$ admits a representative $\hat{\omega} \in \Lambda_{\sigma^k}^p(J^k(E))$ for some k = 0, 1, 2, . . . and $\sigma^k = (\pi_k^\infty)^* \hat{\omega}$. We call k the order of ω. If $X^1, X^2, ..., X^p$ are tangent vectors to $J^\infty(E)$ at σ then, by definition,

$$\omega(X^1, X^2, ..., X^p) = \hat{\omega}\left((\pi_k^\infty)^* X^1, (\pi_k^\infty)^* X^2, ..., (\pi_k^\infty)^* X^p\right).$$

Observe that this is well-defined, that is independent of the choice of representative $\hat{\omega}$ of ω. Evidently, if ω is of order k and one of the vector fields $X^1, X^2,..., X^p$ is π_k^∞ vertical, then $\omega(X^1, X^2,..., X^p) = 0$.

A section of $\Lambda^p(J^k(E))$ is a differential p -form on $J^k(E)$.We denote the vector space of all differential forms on $J^k(E)$ by $\Omega^p(J^k(E))$.These spaces of differential p forms also constitute a direct limit system whose direct limit is the vector space of all differential p forms on $J^\infty(E)$ and is denoted by $\Omega^p(J^\infty(E))$.Again, every smooth differential p form ω on $J^\infty(E)$ is represented by a p form $\hat{\omega}$ on $J^k(E)$ for some k. In local coordinates (x,u,U) a p-form ω on $J^\infty(U)$ is therefore a finite sum of terms of the type

$$A[x,u]du_{I_1}^{\alpha_1} \Lambda du_{I_2}^{\alpha_2} \Lambda....\Lambda du_{I_a}^{\alpha_a} \Lambda dx^{i_1} \Lambda dx^{i_2} \Lambda...\Lambda dx^{i_b} \quad \textbf{(5.1)}$$

where a + b = p and where the coefficient A is a smooth function on $J^\infty(U)$.The order of the term **(5.1)** is the maximum of the orders of the coefficient function

A[x, u] the differentials $du_{I_l}^{\alpha}$.

If $\hat{\omega}$ is a p -form on $J^k(E)$ and $X^1, X^2,..., X^p$ are generalized vector fields on $J^k(E)$ of type $m_1, m_2,.., m_p$ respectively, then the function $\hat{\omega}(X^1, X^2,..., X^p)$ is a smooth function on $J^\infty(E)$ the order of which is equal to the maximum of $m_1, m_2,.., m_p$. If ω is a differential form on $J^\infty(E)$ which is represented by a form $\hat{\omega}$ on $J^k(E)$ and $X^1, X^2,..., X^p$ are vector fields on $J^\infty(E)$ represented by

sequences of generalized vector fields $\{X_l^1\}, \{X_l^2\},..., \{X_l^p\}$ for $l = 0,1,2,...$, then

$$\omega(X^1, X^2,..., X^p) = \hat{\omega}\left(X_k^1, X_k^2,..., X_k^p\right)$$

With these definitions in hand, much of the standard calculus of differential forms on finite dimensional manifolds readily extends to the infinite jet bundle.

Let ω be a differential p form on $J^\infty(E)$ which is represented by the form $\hat{\omega}$ on $J^k(E)$.If X is a vector field of type $(m_0, m_1, m_2,...)$ on $J^\infty(E)$ which is represented by the sequence of generalized vector fields on $J^k(E)$, and if $\Phi : J^\infty(E) \to J^\infty(F)$ is a smooth map represented by the sequence of maps $\Phi_k^{m_k} : J^{m_k}(E) \to J^k(F)$ and ω is a form on $J^\infty(F)$ represented by a form $\hat{\omega}$ on $J^k(F)$ then the pullback form $\Phi^*(\omega)$ is represented by the form $\Phi_k^{m_k})^*(\hat{\omega})$ of order m_k Exterior differentiation

$$d : \Omega^p(J^\infty(E)) \to \Omega^{p+1}\left(J^\infty(E)\right)$$

is similarly defined via representatives if ω is a p form on $J^\infty(E)$ represented by $\hat{\omega}$ on $J^k(E)$,then $d\omega$ is the $p+1$ form on $J^\infty(E)$ represented by $d\hat{\omega}$. In local coordinates, the differential df of a function of order k is given by

$$df = \frac{\partial f}{\partial x^i} dx^i + (\partial_\alpha f) du^\alpha + (\partial_\alpha^i f) du_i^\alpha + ... + (\partial_\alpha^{i_1 i_2 .. i_k} f) du_{i_1 i_2 .. i_k}^\alpha$$

$$= \frac{\partial f}{\partial x^i} dx^i + \sum_{|I|=0}^{k} \left(\partial_\alpha^I f\right) du_I^\alpha . \qquad \textbf{(5.2)}$$

When the order of f is unspecified, we simply extend the summation in **(5.2)** from

$|I| = k$ to $|I| = \infty$ and bear in mind that sum is indeed a finite one.

Let X and Y are vector fields on $J^\infty(E)$ and suppose that ω is a one form. It follows from the above definitions and the invariant definition of the exterior derivative d on finite dimensional manifolds, that

$$(df)(X) = X(f),$$

$$(d\omega)(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega[(X,Y)],$$

and so on.

Lie differentiation of differential forms on the infinite jet bundle is exceptional in this regard. This is due to the fact that for an arbitrary vector field X on $J^\infty(E)$, there is no general existence theorem for the integral curves of X and hence even the

Short time flow of X may not be defined. However, when X is a projectable vector field on $J^\infty(E)$, then the flow of each projection X_k is a well-defined local diffeomorphism $\phi_k(t)$ on $J^k(E)$ for each k. If ω is represented by the form $\hat{\omega}$ on $J^k(E)$, define

$$[L_X(\omega)](\sigma) = [L_{X_k}(\hat{\omega})](\sigma^k) = \left[\frac{d}{dt}\left[(\phi_k(t))^*(\hat{\omega})\right](\sigma)\right]_{t=0}.$$

From this definition, it can be proved that for vector fields $X_1, X_2, \ldots, X_p$,

$$L_X\omega(X_1, X_2, \ldots, X_p) = X\left(\omega(X_1, X_2, \ldots, X_p)\right) + \sum_{i=1}^{p} (-1)^{i+1}\omega\left([X, X_i], X_1, \ldots, \hat{X}_i, \ldots, X_p\right).$$

(5.3)

For a non-projectable vector field X the right-hand side of this equation is still a well-defined derivation on $\Omega^p(J^\infty(E))$ and so, for such vector fields, we simply adopt **(5.3)** as the definition of Lie differentiation.

Now let $\Omega^*(J^\infty(E))$ be the full exterior algebra of differential forms on $J^\infty(E)$ .The contact ideal $C(J^\infty(E))$ is the ideal in $\Omega^*(J^\infty(E))$ of forms ω such that for all $\sigma \in J^\infty(E)$ and local sections s of E around $\sigma^0 = \pi_E^\infty(\sigma)$,

$$\left[j^\infty(s)\right]^*(x)\omega(\sigma) = 0.$$

If $\omega \in C$, then $d\omega \in C$ so that C is actually a differential ideal.

A local basis for C is provided by the contact one forms

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ISSN: 2663-5798
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$$\theta_I^{\alpha} = du_I^{\alpha} - u_{I_j}^{\alpha} dx^j ,$$

where |I| = 0, 1, 2, ... We call |I| the order of the contact form θ_I^{α} even though this form is defined on the (|I|+1)- st jet bundle over U.

Definition (5.4)

If $\pi : U \to U_0$ is a local coordinate neighborhood for E and $\Xi : U_0 \to J^{\infty}(U)$ satisfies

$\Xi^*(\omega) = 0$

for all $\omega \in C$, then there exists a local section $s : U_0 \to U$ such that

$\Xi(x) = j^{\infty}(s)(x)$

for all $x \in U_0$.

Proposition (5.5)

Let $\pi : E \to M$ and $\rho : F \to N$ be two fibered manifolds and let $\phi : E \to F$ be a smooth map which covers a local diffeomorphism $\phi_0 : M \to N$.

(i) The prolongation of ϕ, $pr\phi : J^{\infty}(E) \to J^{\infty}(F)$ preserves the ideal of contact forms, i.e.,

$$[pr\phi]^* C(J^{\infty}(F)) \subset C(J^{\infty}(E))$$

(ii) Let $\Phi : J^{\infty}(E) \to J^{\infty}(F)$ be a smooth map which covers ϕ . If Φ preserves the contact ideal, then $\Phi = pr\phi$.

Proof:

To prove (i), let $\omega \in C(J^{\infty}(F))$ and let, $\eta = [pr\phi]^*(\omega)$ .We show that, $\eta \in C(J^{\infty}(E))$.Let $\sigma = j^{\infty}(s)(x)$ be a point in $J^{\infty}(E)$ where s is a local section of E around x and let $\tilde{s} = \phi \circ s \circ \phi_0^{-1}$ be the induced local section of F around the point $y = \phi_0(x)$.Let $\tilde{\sigma} = j^{\infty}(\tilde{s})(y)$.The definition **(3.5)** of prϕ implies that

$$pr\phi \circ j^{\infty}(s) = j^{\infty}(\tilde{s}) \circ \phi_0 .$$

The chain rule now gives

$$[j^{\infty}(s)]^*(x)\eta(\sigma) = [j^{\infty}(s)]^*(x)\left([pr\phi]^*(\sigma)\omega(\tilde{\sigma})\right)$$

$$= [pr\phi \circ j^{\infty}(s)]^*(x)[\omega(\tilde{\sigma})]$$

$$= [j^{\infty}(\tilde{s}) \circ \phi_0]^*(x)[\omega(\tilde{\sigma})]$$

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$$= \phi_0^*(x)\left(\left[j^\infty(\tilde{s})\right]^*(y)\omega(\tilde{\sigma})\right).$$

This last expression vanishes since ω lies in the contact ideal of $J^\infty(F)$. Therefore η belongs to the contact ideal of $J^\infty(E)$.

To prove (ii), let $\pi : U \to U_0$ and $\rho : V \to V_0$ be coordinate neighborhoods on E and F such that $\phi_0 : U_0 \to V_0$ is a diffeomorphism. Let $s : U_0 \to U$ be any local section and let $\Xi : V_0 \to J^\infty(V)$ be defined by

$$\Xi(y) = \left(\Phi \circ j^\infty(s) \circ \phi_0^{-1}\right)(y).$$

Because Φ is assumed to preserve the contact ideal, $\Xi^*(\omega) = 0$ for any $\omega \in C(J^\infty(F))$ .This implies that there is a section $\tilde{s} : V_0 \to V$ such that $\Xi(y) = j^\infty(\tilde{s})(y)$ for all $y \in V_0$, i.e.,

$$\Phi \circ j^\infty(s) \circ \phi_0^{-1} = j^\infty(\tilde{s}).$$

Since Φ covers ϕ, it follows immediately that $\tilde{s} = \phi \circ s \circ \phi_0^{-1}$ and hence $\Phi = pr\phi$, as required.

6. The Variational Bicomplex

The theory of variational bicomplexes can be regarded as the natural geometrical setting for the calculus of variations. The geometric objects which appear in the calculus of variations find a place on the vertices of a variational bicomplex [6], and are linked by the morphisms of the bicomplex. Such morphisms are closely related to the differential of forms.

This section is devoted to a detailed analysis of the variational bicomplex for the trivial bundle

$$E : R^n \times R^m \to R^n.$$

Local Exactness and the Homotopy Operators for the Variational Bicomplex

Let E be the trivial bundle $E : R^n \times R^m \to R^n$. Let $\Omega^{r,s} = \Omega^{r,s}(J^\infty(E))$. We shall proof the local exactness of the variational bicomplex by establishing the following three propositions.

Proposition (6.1)

For each $r = 0,1,2,\ldots,n$, the vertical complex

$$0 \to \Omega_M^r \xrightarrow{(\pi_M^\infty)^*} \Omega^{r,0} \xrightarrow{d_V} \Omega^{r,1} \xrightarrow{d_V} \Omega^{r,2} \to \textbf{(6.4)}$$

is exact.

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Proposition (6.2)

For each $s \geq 1$, the augmented horizontal complex

$$0 \to \Omega^{0,s} \xrightarrow{d_H} \Omega^{1,s} \xrightarrow{d_H} \dots \xrightarrow{d_H} \Omega^{n-1,s} \xrightarrow{d_H} \Omega^{n,s} \xrightarrow{I} F^s \to 0 \quad \textbf{(6.5)}$$

is exact.

Proposition (6.3)

The Euler-Lagrange complex $\varepsilon^*(J^\infty(E))$

$$0 \to R \xrightarrow{d_H} \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \dots$$

$$\xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} .\Omega^{n,0} \xrightarrow{E} F^1 \xrightarrow{\delta_V} F^2 \xrightarrow{\delta_V} F^3 \dots . \qquad \textbf{(6.6)}$$

is exact.

Proof of Proposition (6.1)

The exactness (in fact, global exactness) of **(6.4)** at s = 0 has already been established in Proposition**:** (Let $\omega \in \Omega^{r,0}(J^\infty(E))$. Then $d_V \omega = 0$ if and only if ω is the pullback, by π_M^∞ of a r form on M).

For $s \geq 1$, the proof of exactness proceeds along the very same lines as the proof, of the local exactness of the de Rham complex [1] as found in, for example. Let

$$R = u^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \textbf{(6.7)}$$

be the vertical radial vector field on E. Then the prolongation of R is the radial vector field

$$prR = u^\alpha \partial_\alpha + u_i^\alpha \partial_\alpha^i + u_{ij}^\alpha \partial_\alpha^{ij} + \dots,$$

on $J^\infty(E)$ and the corresponding flow on $J^\infty(E)$ is the one parameter family of diffeomorphism

$$\Phi_\varepsilon[x,u] = [x, e^\varepsilon u] = \left(x^i, e^\varepsilon u^\alpha, e^\varepsilon u_i^\alpha, e^\varepsilon u_{ij}^\alpha, \dots\right).$$

Let ω be a type (r, s) form on $J^\infty(E)$ .Then the Lie derivative formula gives

$$\frac{d}{d\varepsilon}\left[\Phi_\varepsilon^* \omega\right] = \Phi_\varepsilon^*\left[L_{pr} R, \omega\right]$$

$$= d_V\left[\Phi_\varepsilon^*(prR \neg \omega)\right] + \Phi_\varepsilon^*\left[prR \neg d_V \omega\right].$$

In this equation we replace ε by log t and integrate the result from t = 0 to t = 1 to arrive at

$$\omega = d_V\left[h_V^{r,s}(\omega)\right] + h_V^{r,s+1}(d_V \omega), \qquad \textbf{(6.8)}$$

where the vertical homotopy operator

$$h_V^{r,s} : \Omega^{r,s} \to \Omega^{r,s-1}$$

is defined by

$$h_V^{r,s}(\omega) = \int_0^1 \frac{1}{t} \Phi^*_{\log t}(prR\neg\omega)dt \,. \quad \textbf{(6.9)}$$

Note that the integrand is a actually smooth function at t = 0. Indeed, let $\omega[x,tu]$ denote the form obtained by evaluating the coefficients of ω at the point $[x,tu]$. For instance, if f is a real-valued function on $J^\infty(E)$ and

$$\omega = f[x,u]\gamma \,,$$

where γ is the wedge product of r of the horizontal forms dx^i and s of the vertical forms θ_I^α , then

$$\omega[x,tu] = f[x,tu]\gamma$$

even though the contact forms θ_I^α contain an explicit u_I^α dependence. With this convention, the integrand in **(6.9)** becomes

$$\left(\frac{1}{t}\Phi^*_{\log t}(prR\neg\omega)\right)[x,u] = t^{s-2}(prR\neg\omega)[x,tu] = t^{s-1}prR\neg\omega[x,tu]. \quad \textbf{(6.10)}$$

Because $s \geq 1$, this is certainly a smooth function of t.

To prove Proposition 6.2, we need the following identity. Recall that the inner Euler operators F_α^I were defined by that D_j is the total vector field $D_j = tot\dfrac{\partial}{\partial x^j}$.

Lemma (6.11)

Let $\omega \in \Omega^{r,s}$ and set $\omega_j = D_j \neg \omega$. Then

$$(|I|+1)F_\alpha^{I_j}(d_H\omega) = F_\alpha^{I_j}(dx^j \wedge \omega) + |I| F_\alpha^{jI'}(dx^i \wedge \omega). \qquad \textbf{(6.12)}$$

Proof of Proposition (6.2)

For $s \geq 1$, the horizontal homotopy operator

$h_H^{r,s} : \Omega^{r,s} \to \Omega^{r-1,s}$

is defined by

$$h_H^{r,s}(\omega)=\frac{1}{s}\sum_{|I|=0}^{k-1}\frac{|I|+1}{n-r+|I|+1}D_I\left[\theta^\alpha\wedge F_\alpha^{I_j}(\omega_j)\right]. \quad \textbf{(6.13)}$$

Let ω be an k^{th} order form of type (r, s). To verify that

$$h_H^{r+1,s}(d_H\omega)+d_H\left[h_H^{r,s}(\omega)\right]=\omega\,, \quad \textbf{(6.14)}$$

for $s\geq 1$ and $1\leq r\leq n$, we multiple **(6.12)** by $\dfrac{1}{s(n-r+|I|}\theta^\alpha$, apply the differential operator D_I and sum on |I|, we have that

$$\delta\omega=\sum_{|I|=0}^{k}D_I\left[\theta^\alpha\wedge F_\alpha^I(\omega)\right],$$

so that the result of this calculation reduces to (6.14).

Equation **(6.14)** also holds for r = 0 (with the understanding that $\Omega^{-1,s}=0$) since $D_j\neg\omega=0$ for any $\omega\in\Omega^{0,s}$. With $r=n$, $h_H^{n,s}(\omega)$ coincides with the form η as given by $\eta=h_H^{n,s}(\omega)$ and consequently we can rewrite the equation $\omega=I(\omega)+d_H(\eta)$ as

$$I(\omega)+d_H\left[\left(h_H^{n,s}(\omega)\right)\right]=\omega\,. \quad \textbf{(6.15)}$$

Together equations **(6.14)** and **(6.15)** prove the exactness of the horizontal augmented horizontal complex **(6.5)**

In the next lemma we use the Lie-Euler operators E_α^I.

Lemma (6.16)

Let $\omega\in\Omega^{r,0}$ be a horizontal, type $(r,0)$ form and let Y be an evolutionary vector field on E. Then, for $r\leq n$

$$L_{prY}\omega=I_Y^{r+1}(d_H\omega)+d_H\left(I_Y^{r+1}(\omega)\right), \quad \textbf{(6.17)}$$

where $I_Y^I:\Omega^{r,0}\to\Omega^{r-1,0}$ is defined by

$$I_Y^r(\omega)=\sum_{|I|=0}^{k}\frac{|I|+1}{n-r+|I|+1}D_I\left[Y^\alpha\wedge E_\alpha^{I_j}(\omega_j)\right]. \quad \textbf{(6.18)}$$

For $r=n$,

$$L_{prY}\omega=prY\neg E(\lambda)+d_H\left(I_Y^n(\omega)\right). \quad \textbf{(6.19)}$$

Proof of Proposition (6.3)

It is possible to prove Proposition **(6.3)** from Propositions **(6.1)** and **(6.2)** using elementary spectral sequence arguments (see [1]).

7. Cohomology of the Variational Bicomplex

In this section we explore some of the global aspects of the variational bicomplex on the infinite jet bundle $J^{\infty}(E)$ of the fibered manifold $\pi : E \to M$ by proving that the interior rows of the augmented variational bicomplex are globally exact.

Definition (7.1)

The sequence of spaces and maps

$$0 \to R \to F[u] \xrightarrow{Grad} V[u] \xrightarrow{Curl} V[u] \xrightarrow{Div} F[u] \xrightarrow{E} F[u] \xrightarrow{H} D[u] \quad (7.2)$$

is a cochain complex - the composition of successive maps is zero. One of the maps in (7.2) is the Euler-Lagrange operator E and for this reason we call this sequence the Euler-Lagrange complex.

Theorem (7.3)

Let $\pi : E \to M$ be a fibered manifold. Then, for each $s \geq 1$, the augmented horizontal complex

$$0 \to \Omega^{0,s}(J^{\infty}(E)) \xrightarrow{d_H} \Omega^{1,s}(J^{\infty}(E)) \xrightarrow{d_H} \Omega^{2,s}(J^{\infty}(E)) \xrightarrow{d_H} \ldots.$$

$$\xrightarrow{d_H} \Omega^{n,s}(J^{\infty}(E)) \xrightarrow{I} F^{s}(J^{\infty}(E)) \to 0 \qquad \textbf{(7.4)}$$

is exact.

Proof:

The exactness of **(7.4)** at $\Omega^{r,s}(J^{\infty}(E))$ is established by using a standard partition of unity argument together with induction on r (see [1]).

CONCLUSION

In mathematics, the Lagrangian theory of fiber bundles is globally formulated in a algebric terms of the variational bicomplex, without appealing to the calculus of variations. For instance, this is the case of classical field theory of fiber bundles

The variational bicomplex is a cochain comlex of the differential graded algebra of exterior forms on jet manifolds of sections of a fiber bundle, Lagrangian and Euler Lagrange operators on a fiber bundle are defined as elements of this bicomplex. Cohomology of the variational bicomplex leads to the global first variational formula and first Noether's theorem.

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الملخص:

فى هذه الورقة العلمية قمنا بتقديم التغايرالمركب (العقدى) المزدوج لعديدة الطيات الليفية و حزمة الشعب وتطوير ضروري لحسبان الحقول المتجهة والصيغ التفاضلية على الحزمة المفصلية ذات الرتبة اللانهائية فى تلك الفضاءات . لقد توجهنا بإنتباهنا خاصة للحقول المتجهة المعممة .

الكلمات المقترحة: متعدد الطيات، الشعب، حزم مفصلية، حقول متجهة، متغير ثنائي المركب.