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# "Calculus of Variations and the Euler Lagrange Equation"

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## ABSTRACT

In this paper we review some typical problems in the calculus of variations that are easy to model, although perhaps not so easy to solve. We also, discuss the basics concepts of the Calculus of Variations and, in particular, consider some applications. This will provide us with the mathematical language necessary for formulating the Lagrangian Mechanics.

**Keywords:** Functional, Variational Problem, Lagrangian Formalism.

## 1. INTRODUCTION

The Calculus of Variations is the art to find optimal solutions and to describe their essential properties. In daily life one has regularly to decide such questions as which solution of a problem is best or worst; which object has some property to a highest or lowest degree; what is the optimal strategy to reach some goal. For example one might ask what is the shortest way from one point to another, or the quickest connection of two points in a certain situation. The isoperimetric problem, already considered in antiquity, is another question of this kind. Here one has the task to find among all closed curves of a given length the one enclosing maximal area. The appeal of such optimum problems consists in the fact that, usually, they are easy to formulate and to understand, but much less easy to solve. For this reasons the calculus of variations or, as it was called in earlier days, the isoperimetric method has been a thriving force in the development of analysis and geometry.

The calculus of variations is concerned with finding extrema and, in this sense; it can be considered a branch of optimization. The problems and techniques in this branch, however, differ markedly from those involving the extrema of functions of several variables owing to the nature of the domain on the quantity to be optimized. A functional is a mapping from a set of functions to the real numbers. The calculus of variations deals with finding extrema for functionals as opposed to functions. The candidates in the competition for an extremum are thus functions as opposed to vectors in $R^n$, and this gives the subject a distinct character. The functionals are generally defined by definite integrals; the sets of functions are often defined by boundary conditions and smoothness requirements, which arise in the formulation of the problem/model.

The calculus of variations is nearly as old as the calculus, and the two subjects were developed somewhat in parallel. In 1927 Forsyth (Bruce (2004) noted that the subject "attracted a rather fickle attention at more or less isolated intervals in its growth." In the eighteenth century, the Bernoulli brothers, Newton, Leibniz, Euler, Lagrange, and Legendre contributed to the subject (Bruce (2004)**,** and their work was extended significantly in the next century by Jacobi and Weierstrass (Bruce (2004). Hilbert, in his renowned 1900 lecture to the International Congress of Mathematicians, outlined 23 (now famous) problems for mathematicians. His 23rd problem (Bruce (2004) is entitled further development of the methods of the calculus of variations.

Hilbert's lecture perhaps struck a chord with mathematicians. In the early twentieth century Hilbert, Noether, Tonelli, Lebesgue, and Hadamard (Bruce (2004)) among others made significant contributions to the field. Although by Forsyth's time the subject may have "attracted rather fickle attention," many of those who did pay attention are numbered among the leading mathematicians of the last three centuries.

## 2. Functionals (Some simple Variational Problems)

Variable quantities called functionals play an important role in many problems arising in analysis, mechanics, geometry, etc. By a functional, we mean a correspondence which assigns a definite (real) number to each function (or curve) belonging to some class. Thus, one might say that a functional is a kind of function, where the independent variable is itself a function (or curve). The following are examples of functionals:

1 - Consider the set of all rectifiable plane curves. A definite number is associated with each such curve, namely, its length. Thus, the length of a curve is a functional defined on the set of rectifiable curves.

2- Suppose that each rectifiable plane curve is regarded as being made out of some homogeneous material. Then if we associate with each such curve the ordinate of its center of mass, we again obtain a functional.

3- Consider all possible paths joining two given points A and B in the plane. Suppose that a particle can move along any of these paths, and let the particle have a definite velocity $v(x, y)$ at the point $(x, y)$.Then we obtain a functional by associating with each path the time the particle takes to traverse the path.

4- Let $y(x)$ be an arbitrary continuously differentiable function, defined on the interval $[a,b]$. Then the formula

$$J[y] = \int_a^b y'^2(x)dx$$

defines a functional on the set of all such functions $y(x)$.

5- As a more general example, let $F(x, y, z)$ be a continuous function of three variables. Then the expression

$$J[y] = \int_a^b F(x, y(x), y'(x))dx, \qquad (1)$$

where $y(x)$ ranges over the set of all continuously differentiable functions defined on the interval $[a,b]$, defines a functional. By choosing different functions $F(x, y, z)$, we obtain different functionals. For example, if

$$F(x, y, z) = \sqrt{1+z^2},$$

$J[y]$ is the length of the curve $y = y(x)$, as in the first example, while if

$$F(x, y, z) = z^2,$$

$J[y]$ reduces to the case considered in the fourth example. In what follows, we shall be concerned mainly with functionals of the form (1).

Particular instances of problems involving the concept of a functional were considered more than three hundred years ago, and in fact, the first important results in this area are due to Euler (1707 -1783) (Gel'fand 1963). Nevertheless, up to now, the "calculus of functionals" still does not have methods of a generality comparable to the methods of classical analysis (i.e., the ordinary " calculus of functions"). The most developed branch of the "calculus of functionals" is concerned with finding the maxima and minima of functionals, and is called the "calculus of variations." Actually, it would be more appropriate to call this subject the "calculus of variations in the narrow sense," since the significance of the concept of the variation of a functional is by no means confined to its applications to the problem of determining the extrema of functionals.

We now indicate some typical examples of variational problems, by which we mean problems involving the determination of maxima and minima of functionals.

1- Find the shortest plane curve joining two points A and B, i.e., find the curve $y = y(x)$ for which the functional

$$\int_a^b \sqrt{1+y'^2}\,dx$$

achieves its minimum. The curve in question turns out to be the straight line segment joining A and B.

2- The following variational problem, called the isoperimetric problem, was solved by Euler (Bruce (2004)**:** Among all closed curves of a given length $l$, find the curve enclosing the greatest area. The required curve turns out to be a circle.

All of the above problems involve functionals which can be written in the form

$$\int_a^b F(x, y, y')dx\,.$$

Such functionals have a "localization property" consisting of the fact that if we divide the curve $y = y(x)$ into parts and calculate the value of the functional for each part, the sum of the values of the functional for the separate parts equals the value of the functional for the whole curve. It is just these functionals which are usually considered in the calculus of variations.

An important factor in the development of the calculus of variations was the investigation of a number of mechanical and physical problems, e.g., the brachistochrone problem (mentioned below). In turn, the methods of the calculus of variations are widely applied in various physical problems. It should be emphasized that the application of the calculus of variations to physics does not consist merely in the solution of individual, albeit very important problems.

## 3. The Brachistochrone Problem (Filip (2018).

In June 1696 Johann Bernoulli published the description of a mathematical problem in the journal Acta Eruditorum (Filip (2018). Bernoulli also sent a letter containing the problem to Leibniz on 9 June 1696, who returned his solution only a few days later on 16 June, and commented that the problem tempted him “like the apple tempted Eve”. Newton also published a solution (after the problem had reached him) without giving his identity, but Bernoulli identified him “ex ungue leonem” (from Latin, “by the lion’s claw”) (Filip (2018).

The problem that the great minds of the time found so irresistible was formulated as follows:

Given two points A and B in a vertical [meaning “not horizontal”] plane, one shall find a curve AMB for a movable point M, on which it travels from the point A to the other point B in the shortest time, only driven by its own weight.

The resulting curve is called the brachistochrone (from Ancient Greek, “shortest time”) curve.

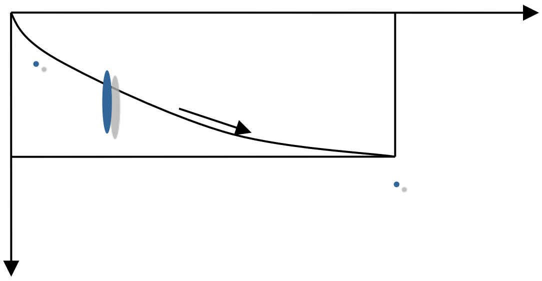

**Fig.1 slide curves from the origin to $(\bar{x}, \bar{y})$.**

Amore precise formulation of the brachistochrone problem is as follows: We look for the curve connecting the origin $(0,0)$ to the point $(\bar{x},\bar{y})$, where $\bar{x}>0$, $\bar{y}<0$ such that under the gravitational acceleration (in the negative y-direction) a point mass $m>0$ slides from rest at $(0,0)$ to $(\bar{x},\bar{y})$ quickest among all such curves (Figure **1**). We parameterize a point $(x, y)$ on the curve by the time $t \geq 0$ that the mass takes to reach it. The sliding point mass has kinetic and potential energies

$$E_{Kin} = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = \frac{m}{2}\left(\frac{dx}{dt}\right)^2\left[1+\left(\frac{dy}{dx}\right)^2\right],$$

$$E_{Pot} = mgy,$$

where $g \approx 9.81 m/s^2$ is the gravitational acceleration on Earth. The total energy $E_{Kin} + E_{Pot}$ is zero at the beginning and conserved along the path. Hence,

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = -mgy.$$

We can solve this for $dt/dx$ (where $t = t(x)$ is the inverse of the x-parameterization) to get

$$\frac{dt}{dx} = \sqrt{\frac{1+(y')^2}{-2gy}}, \qquad \left(\frac{dt}{dx}\right) \geq 0,$$

where we wrote $y' = \frac{dy}{dx}$. Integrating over the whole x-length along the curve from

$0$ to $\bar{x}$, we get for the total slide duration T [y] that

$$\mathrm{T}[y] = \frac{1}{\sqrt{2g}}\int_0^{\bar{x}}\sqrt{\frac{1+(y'(x))^2}{y(x)}}dx.$$

## 4. Partial and Total Differentiation

(a) If $u = f(x, y, \ldots, z)$, $x = x(r, s, \ldots, t)$, $y = y(r, s, \ldots, t), \ldots,$ $z = z(r, s, \ldots, t)$, then

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \ldots + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r},$$

where $r$ may successively be replaced by $s,...,t$.

(b) If $u = f(x, y,..., z, t)$, $x = x(t), y = y(t),..., z = z(t)$, then

$$\frac{du}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + ... + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

(c) The quantity $p(x, y) + q(x, y)y'$ - where the prime indicate ordinary differentiation with respect to $x$ - is the derivative $(dg/dx)$ of some function $g(x, y)$ if and only if $(\partial p/\partial y) = (\partial q/\partial x)$. In this event $p = (\partial q/\partial x), q = (\partial q/\partial y)$.

To understand the basic meaning of the problems and methods of the calculus of variations, it is very important to see how they are related to problems of classical analysis, i.e., to the study of functions of $n$ variables.

## 5. Functions of several variables

In the study of functions of $n$ variables, it is convenient to use geometric language, by regarding a set of $n$ numbers $(y_1,..., y_n)$ as a point in an $n$-dimensional space. In just the same way, geometric language is useful when studying functionals. Thus, we shall regard each function $y(x)$ belonging to some class as a point in some space and spaces whose elements are functions will be called function spaces.

In the study of functions of a finite number $n$ of independent variables, it is sufficient to consider a single space, i.e., $n$-dimensional Euclidean space $R^n$. However, in the case of function spaces, there is no such "universal" space. In fact, the nature of the problem under consideration determines the choice of the function space. For example, if we are dealing with a functional of the form

$$\int_a^b F(x, y, y')dx,$$

it is natural to regard the functional as defined on the set of all functions with a continuous first derivative, while in the case of a functional of the form

$$\int_a^b F(x, y, y', y'')dx,$$

the appropriate function space is the set of all functions with two continuous derivatives. Therefore, in studying functionals of various types, it is reasonable to use various function spaces.

The concept of continuity plays an important role for functionals, just as it does for the ordinary functions considered in classical analysis.

## 6. The Variation of a Functional

I n this section, we introduce the concept of the variation (or differential ) of a functional, analogous to the concept of the differential of a function of $n$ variables. The concept will then be used to find extrema of functionals. First, we give some preliminary facts and definitions.

### (6.1) Definition

Given a normed linear space $R$, let each element $h \in R$ be assigned a number $\varphi[h]$, i.e. , let $\varphi[h]$ be a functional defined on $R$ .Then $\varphi[h]$ is said to be a (continuous) linear functional if

1. $\varphi[\alpha h] = \alpha[\varphi h]$ for any $h \in R$ and any real number $\alpha$ ;

2. $\varphi[h_1 + h_2] = \varphi[h_1] + \varphi[h_2]$ for any $h_1, h_2 \in R$;

3. $\varphi[h]$ is continuous (for all $h \in R$).

### (6.2) Example

If we associate with each function $h(x) \in \wp(a,b)$ its value at a fixed point $x_0$ in $[a,b]$ i.e., if we define the functional $\varphi[h]$ by the formula

$$\varphi[h] = h(x_0),$$

then $\varphi[h]$ is a linear functional on $\wp(a,b)$.

### (6.3) Example

The integral

$$\varphi[h] = \int_a^b h(x)dx,$$

defines a linear functional on $\wp(a,b)$.

### (6.4) Example

The integral

$$\varphi[h] = \int_a^b \alpha(x)h(x)dx,$$

where $\alpha(x)$ is a fixed function in $\wp(a,b)$, defines a linear functional on $\wp(a,b)$.

### (6.5) Lemma

If $\alpha(x)$ is continuous in [a, b], and if

$$\int_a^b \alpha(x)h(x)dx = 0$$

for every function $h(x) \in \wp(a,b)$ such that $h(a) = h(b) = 0$, then $\alpha(x) = 0$ for all $x$ in [a, b].

**Proof of lemma:** (Gel'fand 1963)

We now introduce the concept of the variation (or differential) of a functional. Let $J[y]$ be a functional defined on some normed linear space, and let

$$\Delta J[h] = J[y+h] - J[y]$$

be its increment, corresponding to the increment $h = h(x)$ of the " independent variable " $y = y(x)$. If $y$ is fixed, $\Delta J[h]$ is a functional of $h$, in general a nonlinear functional. Suppose that

$$\Delta J[h] = \varphi[h] + \varepsilon\|h\|,$$

where $\varphi[h]$ is a linear functional and $\varepsilon \to 0$ as $\|h\| \to 0$. Then the functional $J[y]$ is said to be differentiable, and the principal linear part of the increment $\Delta J[h]$, i.e., the linear functional $\varphi[h]$ which differs from $\Delta J[h]$ by an infinitesimal of order higher than 1 relative to $\|h\|$, is called the variation (or differential) of $J[y]$ and is denoted by $\delta J[h]$.

Next, we use the concept of the variation (or) differential of a functional to establish a necessary condition for a functional to have an extremum.

### (6.6) Theorem

A necessary condition for the differentiable functional $J[y]$ to have an extremum for $y = \hat{y}$ is that its variation vanish for $y = \hat{y}$, i.e., that

$$\delta J[h] = 0$$

for $y = \hat{y}$ and all admissible $h$.

### Proof

To be explicit, suppose $J[y]$ has a minimum for $y = \hat{y}$.

According to the definition of the variation $\delta J[h]$, we have

$$\Delta J[h] = \delta J[h] + \varepsilon\|h\|, \quad (2)$$

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where $\varepsilon \to 0$ as $\|h\| \to 0$. Thus, for sufficiently small $\|h\|$, the sign of $\Delta J[h]$ will be the same as the sign of $\delta J[h]$. Now, suppose that $\delta J[h_0] \neq 0$ for some admissible $h_0$. Then for any $\alpha > 0$, no matter how small, we have

$$\delta J[-\alpha h_0] = -\delta J[\alpha h_0].$$

Hence, (2) can be made to have either sign for arbitrarily small $\|h\|$. But this is impossible, since by hypothesis $J[y]$ has a minimum for $y = \hat{y}$, i.e.,

$$\Delta J[h] = J[\hat{y} + h] - J[\hat{y}] \geq 0$$

for all sufficiently small $\|h\|$. This contradiction proves the theorem.

## 7. The Lagrangian Formalism

Using the ideas of previous section, we shall now derive necessary conditions which are to be satisfied by critical points of variational integrals. The principal condition is the vanishing of the first variation which, for smooth critical points, implies the Euler equation and, in case of free boundary values, also the natural boundary condition. We shall concentrate our attention to first order variational problems where the Lagrangian only involves the unknown function and its derivatives of first order.

### 7.1. The Simplest Variational Problem. Euler's Equation

We shall consider functionals $F$ of the type

$$F(u) = \int_a^b F(x, y, y')dx, \qquad (3)$$

which will be called variational integrals. We shall write $F_\Omega(u)$, if we want to indicate the domain of integration $\Omega$. The integrand $F(x, y, z)$ of such an integral $F(u)$ will be denoted as Lagrangian, or variational integrand, or Lagrange function.

In other words, the simplest variational problem consists of finding a weak extremum of a functional of the form (3), where the class of admissible curves consists of all smooth curves joining two points. The first two examples on section 2, involving the brachistochrone and the shortest distance between two points, are variational problems of just this type. To apply the necessary condition for an extremum (found in Sec. 7.3) to the problem just formulated, we have to be able to calculate the variation of a functional of the type (3). We now derive the appropriate formula for this variation.

### (7.2) Theorem

Let $J[y]$ be a functional of the form

$$\int_a^b F(x, y, y')dx, \qquad (4)$$

defined on the set of functions $y(x)$ which have continuous first derivatives in [a, b] and satisfy the boundary conditions $y(a) = A, y(b) = B$. Then a necessary condition for $J[y]$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfy Euler's equation

$$F_y - \frac{d}{dx} F_{y'} = 0. \qquad (5)$$

**Proof:** (Gel'fand 1963)

The integral curves of Euler's equation are called extremals. Since Euler's equation is a second-order differential equation, its solution will in general depend on two arbitrary constants, which are determined from the boundary conditions $y(a) = A, y(b) = B$. The problem usually considered in the theory of differential equations is that of finding a solution which is defined in the neighborhood of some point and satisfies given initial conditions (Cauchy's problem). However, in solving Euler's equation, we are looking for a solution which is defined over all of some fixed region and satisfies given boundary conditions. Therefore, the question of whether or not a certain variational problem has a solution does not just reduce to the usual existence theorems for differential equations.

### 7.3. Applying the Euler–Lagrange equation

### (1) We find the solution stationary path of the distance functional

$$J[y] = \int_a^b \sqrt{1 + y'^2}\, dx, \qquad y(a) = A, y(b) = B, \qquad (6)$$

considered in Section 2.

Comparing equations (4) and (6), we see that the integrand is given by

$$F(x, y, y') = \sqrt{1 + y'^2(x)}.$$

To use the Euler–Lagrange equation, we need to calculate $\partial F / \partial y$ and $\partial F / \partial y'$. The first thing to notice is that the integrand depends explicitly only on $y'$ not on $x$ or $y$. Therefore

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} \text{ and } \frac{\partial F}{\partial y} = 0$$

Hence the Euler–Lagrange equation (5) becomes

$$\frac{d}{dx}\left( \frac{y'}{\sqrt{1 + (y')^2}} \right) = 0. \qquad (7)$$

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We could differentiate this to get a second-order differential equation for $y$ , which we could then solve by integration. However, equation (7) is already in a convenient form to be integrated directly.

Integrating both sides of equation (7) with respect to $x$ gives

$$\frac{y'}{\sqrt{1+(y')^2}}=\alpha,$$

for some constant $\alpha$ . Its solution was shown to be

$$y=\frac{B-A}{b-a}x+\frac{Ab-Ba}{b-a}.$$

**(2)** We use the equation $\mathrm{T}[y]=\frac{1}{\sqrt{2g}}\int_0^{\bar{x}}\sqrt{\frac{1+\left(y'(x)\right)^2}{y(x)}}dx,\ y(0)=0, y(\bar{x})=1$ (8) (considered in Section 3) to show that the time taken for a particle of mass $m$ to slide down the curve $y=x/b$ from rest at the origin to the point $(b,1)$ is

$$\mathrm{T}=2\sqrt{\frac{b^2+1}{2g}}.$$

As $y=\frac{x}{b}$, we have $y'=\frac{1}{b}$, so

$$\sqrt{\frac{1+y'^2}{y}}=\sqrt{\frac{1+1/b^2}{x/b}}=\sqrt{\frac{b^2+1}{b}}\frac{1}{\sqrt{x}}.$$

Therefore equation (8) tells us that the time T satisfies

$$\mathrm{T}=\sqrt{\frac{b^2+1}{2gb}}\int_0^b\frac{1}{\sqrt{x}}dx=2\sqrt{\frac{b^2+1}{2gb}}\left[\sqrt{x}\right]_0^b=2\sqrt{\frac{b^2+1}{2g}}.$$

## 8. The Lagrangian Method

Here is the procedure. Consider the following seemingly silly combination of the

kinetic and potential energies (T and V, respectively),

$$L\equiv T-V \qquad (8.1)$$

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This is called the Lagrangian. Yes, there is a minus sign in the definition (a plus sign would simply give the total energy). In the problem of a mass on the end of a spring, $T = \frac{m}{2}\dot{x}^2$ and $V = \frac{k}{2}x^2$, so we have

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 . \qquad (8.2)$$

Now write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} . \qquad (8.3)$$

This equation is called the Euler-Lagrange (E-L) equation. For the problem at hand, we have $\partial L / \partial \dot{x} = m\dot{x}$ and $\partial L / \partial x = -kx$ , so equation (8.3) gives

$$m\ddot{x} = -kx ; \qquad (8.4)$$

which is exactly the result obtained by using F = ma. An equation such as equation (8.4), which is derived from the Euler-Lagrange equation, is called an equation of motion. If the problem involves more than one coordinate, as most problems do, we just have to apply equation (8.3) to each coordinate.

## 8.5. Theorem (Hamilton's Principle)

The motion of a system of particles $q(t)$ from a given initial configuration $q(t_0)$ to a given final configuration $q(t_1)$ in the time interval $[t_0, t_1]$ is such that the functional

$$J(q) = \int_{t_0}^{t_1} L(t, q, \dot{q})\,dt$$

is stationary.

## 8.6. Example (Kepler problem)

The Kepler problem models planetary motion [1]. It is one of the most heavily studied problems in classical mechanics. Keeping with our no frills approach, we consider the simplest problem of a single planet orbiting around the sun, and ignore the rest of the solar system. Assuming the sun is fixed at the origin, the kinetic energy of the planet is

$$\mathrm{T} = \frac{1}{2}m\left(\dot{x}^2(t) + \dot{y}^2(t)\right) = \frac{1}{2}m\left(\dot{r}^2(t) + r^2(t)\dot{\theta}^2(t)\right),$$

where $r$ and $\theta$ denote polar coordinates and $m$ is the mass of the planet. We can deduce the potential energy function $V$ from the gravitational law of attraction

$$f = -\frac{GmM}{r^2} ,$$

where $f$ is the force (acting in the radial direction), $M$ is the mass of the sun, and $G$ is the universal gravitation constant. Given that

$$f = -\frac{\partial V}{\partial r},$$

we have

$$V(r) = -\int f(r)dr = -\frac{GmM}{r};$$

hence,

$$L(r,\theta) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GmM}{r}.$$

Hamilton's Principle implies that the motion of the planet from an initial observation $(r(t_0),\theta(t_0))$ to a final observation $(r(t_1),\theta(t_1))$ is such that

$$J(r,\theta) = \int_{t_0}^{t_1}\left(\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GmM}{r}\right)dt$$

is stationary.

The reader may be wondering about the fate of the constant of integration in the last example. Any potential energy of the form $-GmM/r$ +const. will produce the requisite force $f$ .

## 9. CONCLUSION

The calculus of variations has a long history of interaction with other branches of mathematics such as geometry and differential equations, and with physics, particularly mechanics. More recently, the calculus of variations has found applications in other fields such as economics and electrical engineering. Much of the mathematics underlying control theory, for instance, can be regarded as part of the calculus of variations.

Lagrangian formalism is the main tool of theoretical classical mechanics. Calculus of Variations is apart of Mathematics which Lagrangian formalism is based on.

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## الملخص:

فى هذه الورقة العلمية بالتحديد، قمنا بمراجعة بعض المسائل فى حسبان التغاير سهلة العرض، ربما ليست من السهل حلها. كذلك ناقشنا المفاهيم الأساسية لحسبان التغاير ، وبشكل خاص أخذنا بعين الإعتبار بعض التطبيقات . هذا سوف يكون داعم لنا مع لغة الرياضيات الضرورية لصياغة ميكانيكية لاجرانج.

**الكلمات المقترحة:** دالى، مسألة مشكلة التغاير، صيغة لاجرانج .