$$f(x) = g(x) + \int_0^{h(x)} k(x,t)\, f(t)\,dt$$

“A SOLUTION OF THE VOLTERRA EQUATION BY MACLAURIN SERIES"

$$f(x) = g(x) + \int_0^{h(x)} k(x,t)\, f(t)\,dt$$

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Abstract

In this paper we introduce a series solution for Volterra Equation with limits of integration from 0 to a polynomial which is a generalization of the classical Volterra equation for n= 1.

n=1

Keywords Remove and Phrases: Integral equation, Maclaurin series, Leibniz Rule, Functional-ifferential equations**,** Volterra equation.

***** This paper is considered generalization to the published paper in proceedings of the 9th international conference on advances in computing, communication and information technology-CCIT2019 ISBN No. 978-1-63248-181-8 DOI : 10.15224/978-1-63248-181-8-14 which titel is $f(x) = g(x) + \int_0^{x^n} k(x,t) f(t)dt$

1. INTRODUCTION

In the late 17th century and early 18th century mathematicians worked on the development of different branches of mathematics from geometry to calculus to trigonometry to algebra to number theory to differential and integrative equations ... to solve many problems in mathematics Physics and engineering like Vito Volterra, Fourier, Gottfried Wilhelm,…
At the end of the nineteenth century saw an increasing interest in integral equations, mainly because of their connection with some of the differential equations of mathematical physics. From this work emerged four general forms of integral equations now called Volterra and Fredholm equations of the first and second kinds.

So the most basic type of integral equation is called a Fredholm Equation, where the limits of integration a and b are constants, is given by the form

$$f(x) = g(x) + \int_a^b k(x,t) f(t)dt$$

If one limit of integration is a variable, the equation is called a Volterra equation, is given by the form

$$f(x) = g(x) + \int_a^x k(x,t) f(t)dt$$

Volterra integral equations arise in many scientific and engineering problems a large class of initial and boundary value problems can be converted to Volterra integral equations such as, heat conduction problem, concrete problem of mechanics or physics, diffusion problems, electroelastic, contact problems , plasma physics, the image deblurring problem and its regularization, axisymmetric contact problems for bodies with complex rheology[7].

In this paper, we studied Volterra integral equation with limits of integral from 0 to a polynomial $h(x)$ where

$h(x) = a_3x^3 + a_4x^4 + ...a_nx^n$ and n is a positive integer, is of the form $f(x) = g(x) + \int_0^{h(x)} k(x,t) f(t)dt$,

where $g(x)$ and $k(x)$ are known functions, and $f(x)$ is the unknown function to be found. As far as the authors know, no solution for the case from zero to a polynomial is given.Actually for deg ($h(x)$)= 0, the equation leads to a differential equation (Polyanin and Manzhirov, 2008, Wazwaz, 2015 and Rahman, 2007). While for deg($h(x)$) > 1 the equation leads to a functional differential equation. assume here that the functions $g(x)$, $f(x)$ and $k(x)$ are analytic at 0 So, we can use the Maclaurin series technique to get our solution, namely

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + ..., \qquad (1)$$

So, our process simply, is to determine the derivatives at 0 of $f(x)$ in terms of the derivatives at 0 of $g(x)$ and $k(x)$ to get the solution of

$$f(x) = g(x) + \int_0^{h(x)} k(x,t) f(t)dt \qquad (2)$$

2. Analysis the Method

$$f(x) = g(x) + \int_0^{h(x)} k(x,t) f(t)dt$$

By Leibniz rule find $f'(x), f''(x), f'''(x),....$

$$f'(x) = g'(x) + \int_0^{h(x)} \frac{\partial}{\partial x} k(x,t) f(t)dt + h'(x)k(x,h(x)) f(h(x))$$

$$f''(x) = g''(x) + \int_0^{h(x)} \frac{\partial^2}{\partial x^2} k(x,t) f(t)dt + [h''(x)k(x,h(x)) + 2h'(x)k'(x,h(x))] f(h(x))$$

$$+[(h'(x))^2 k(x,h(x)] f'(h(x))$$

$$f'''(x) = g'''(x) + \int_0^{h(x)} \frac{\partial^3}{\partial x^3} k(x,t) f(t)dt + [h'''(x)k(x,h(x)) + 3h''(x)k'(x,h(x))$$

$$+3h'(x)k''(x,h(x))] f(h(x))$$

$$+[3h'(x)h''(x)k(x,h(x)) + 3(h'(x))^2 k'(x,h(x))] f'(h(x))$$

$$+[3h'(x)h''(x)k(x,h(x)) + 3(h'(x))^2 k'(x,h(x))] f'(h(x))$$

$$+[(h'(x))^3 k(x,h(x))] f''(h(x))$$

$$f^{(4)}(x) = g^{(4)}(x) + \int_0^{h(x)} \frac{\partial^4}{\partial x^4} k(x,t) f(t)dt + [h^{(4)}(x)k(x,h(x)) + 4h'''(x)k'(x,h(x))$$

$$
\begin{aligned}
&+6h''(x)k''(x,h(x))+4h'(x)k'''(x,h(x))]f(h(x))\\
&+[4h'(x)h'''(x)k(x,h(x))+12h'(x)h''(x)k'(x,h(x))+6(h'(x))^2k''(x,h(x))\\
&+3(h''(x))^2k(x,h(x))]f'(h(x))\\
&+[6(h'(x))^2h''(x)k(x,h(x))+4(h'(x))^3k'(x,h(x))]f''(h(x))\\
&+[(h'(x))^4k(x,h(x))]f'''(h(x))\\
&f^{(5)}(x)=g^{(5)}(x)+\int_0^{h(x)}\frac{\partial^5}{\partial x^5}k(x,t)f(t)dt+[h^{(5)}(x)k(x,h(x))+5h^{(4)}(x)k'(x,h(x))\\
&+10h'''(x)k''(x,h(x))+10h''(x)k'''(x,h(x))+5h'(x)k^{(4)}(x,h(x))]f(h(x))\\
&+[5h'(x)h^{(4)}(x)k(x,h(x))+20h'(x)h'''(x)k'(x,h(x))+30h'(x)h''(x)k''(x,h(x))\\
&+10(h'(x))^2k'''(x,h(x))+10h''(x)h'''(x)k(x,h(x))+15(h''(x))^2k'(x,h(x))]f'(h(x))\\
&+[10(h'(x))^2h'''(x)k(x,h(x))+30(h'(x))^2h''(x)k'(x,h(x))+10(h'(x))^3k''(x,h(x))\\
&+15h'(x)(h''(x))^2k(x,h(x))]f''(x)\\
&+[10(h'(x))^3h''(x)k(x,h(x))+5(h'(x))^4k'(x,h(x))]f'''(x)\\
&+[(h'(x))^5k(x,h(x))]f'''(h(x))\\
&\vdots
\end{aligned}
\tag{3}
$$

Now, replacing x by 0 we obtain

$$
\begin{aligned}
&f(0)=g(0)\\
&f'(0)=g'(0)\\
&f''(0)=g''(0)\\
&f'''(0)=g'''(0)+[6a_3k(0,0)]f(0)\\
&f^{(4)}(0)=g^{(4)}(0)+[24a_4k(0,0)+24a_3k'(0,0)]f(0)\\
&f^{(5)}(0)=g^{(5)}(0)+[120a_5k(0,0)+120a_4k'(0,0)+60a_3k''(0,0)]f(0)\\
&\vdots
\end{aligned}
\tag{4}
$$

The general form at $x=0$ is

$$
f^{(m)}(0)=\begin{cases} g^{(m)}(0) & , m<3 \\ g^{(m)}(0)+f(0)\sum_{i=0}^{m-3}p(m,m-i)a_{m-i}k^{(i)}(0,0) & , O.W \end{cases}
$$

Where $f(0)=g(0), a_{m-i}$ the coefficients of $h(x)$, and $p(m,m-i)=m(m-1)....(i+1)$

Applying Maclaurin series

$$f(x)=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+....,$$

So the solution

$$f(x)=\sum_{m=0}^{\infty}[g^{(m)}(0)]\frac{x^m}{m!}+g(0)\sum_{m=3}^{\infty}[\sum_{i=0}^{m-3}p(m,m-i)a_{m-i}k^{(i)}(0,0)]\frac{x^m}{m!}$$

$$=g(x)+g(0)\sum_{m=3}^{\infty}[\sum_{i=0}^{m-3}p(m,m-i)a_{m-i}k^{(i)}(0,0)]\frac{x^m}{m!}+ \quad (5)$$

3.Conclusion

The present study is the first study solving the Volterra Integral Equation where the ***upper limit*** of integral is polynomial. The result generated in this study would be helpful the researchers Volterra Integral Equation with others limt.

4. Applications and numerical results

(1) The solution of the integral equation

$$f(x)=e^{-x}-x^4-x^5+\int_0^{x^3+x^4}(xe^t)f(t)dt,$$

is

$$f(x)=e^{-x}$$

(2) The solution of the integral equation

$$f(x)=x^{10}+x^{11}+e^{x^2}+\int_0^{x^6+x^7}(-x^4e^{t^{-2}})f(t)dt,$$

is

$$f(x)=e^{x^2}$$

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